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# The $TQ$ equation of the eight-vertex model for complex elliptic roots of unity

**Klaus Fabricius**[1] **and Barry M McCoy**[2]

[1] Physics Department, University of Wuppertal, 42097 Wuppertal, Germany
[2] Institute for Theoretical Physics, State University of New York, Stony Brook, NY 11794-384, USA

E-mail: Fabricius@theorie.physik.uni-wuppertal.de and mccoy@max2.physics.sunysb.edu

## Abstract

We extend our studies of the $TQ$ equation introduced by Baxter in his 1972 solution of the eight-vertex model with parameter $\eta$ given by $2L\eta = 2m_1 K + \mathrm{i}m_2 K'$ from $m_2 = 0$ to the more general case of complex $\eta$. We find that there are several different cases depending on the parity of $m_1$ and $m_2$.

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## 1. Introduction

In 1972, Baxter published one of the most important and influential papers in statistical mechanics of the 20th century [1]. A loose statement in this paper is that Baxter 'solves the eight-vertex model'. A more precise statement is that Baxter constructs a matrix $Q(v)$ which satisfies the following functional equation with the transfer matrix $T(v)$ of the eight-vertex model:

$$T(v)Q(v) = [h(v+\eta)]^N Q(v-2\eta) + [h(v-\eta)]^N Q(v+2\eta), \tag{1}$$

with

$$[T(v), Q(v')] = 0, \tag{2}$$

$$[Q(v), Q(v')] = 0. \tag{3}$$

Here, $N$ is the number of horizontal sites, $h(v)$ is a suitable quasiperiodic function and $\eta$ is a constant that is present in the Boltzmann weights of the model which satisfies

$$2L\eta = 2m_1 K + \mathrm{i}m_2 K', \tag{4}$$

where $L$, $m_1$ and $m_2$ are integers, $K$ is the complete elliptic integral of the first kind of modulus $k$ and $K'$ is the complete elliptic integral of the complementary modulus $k' = (1 - k^2)^{1/2}$. The

values of $\eta$ satisfying the condition (4) are referred to in the physics literature as elliptic roots of unity (which we will abbreviate to 'roots of unity'). In the mathematics literature, these values of $\eta$ are referred to as division points.

In the past 35 years, there have been many papers which extend, generalize and comment on equations (1)–(3) which are often collectively referred to as 'the $TQ$ equation'. Some of this literature is by Baxter [2–5], Takhtadzhan and Faddeev [6], Zabrodin [7], Deguchi [8, 9], Bazhanov and Mangazeev [10–12], Roan [13] and the present authors [14–16]. Each paper has added to our understanding of the original paper [1] but each paper leaves some questions unexplored.

The transfer matrix $T(v)$ of the eight-vertex model has two discrete symmetries expressed by the commutation relations

$$[T(v), S] = 0, \qquad [T(v), R] = 0, \tag{5}$$

where (in the canonical basis given in section 2)

$$S = \sigma_1^z \otimes \sigma_2^z \otimes \cdots \otimes \sigma_N^z, \qquad R = \sigma_1^x \otimes \sigma_2^x \otimes \cdots \otimes \sigma_N^x. \tag{6}$$

We note that

$$RS = (-1)^N SR \tag{7}$$

and from (5) it follows that

$$[T(v), RS] = 0. \tag{8}$$

If the transfer matrix $T(v)$ were nondegenerate then the matrix $Q(v)$ of (1)–(3) would also satisfy (5) and (8). However, when (4) holds the matrix $T(v)$ has degenerate eigenvalues and the matrix $Q(v)$ no longer needs to satisfy these commutation relations. For this reason, the matrix $Q$ which satisfies (1)–(3) is not uniquely defined when (4) holds.

The $Q(v)$ matrix can be chosen to have the same discrete symmetries (5) and (8) as the transfer matrix. This choice is made in the 1973 paper of Baxter [2]. We call this choice $Q_{73}(v)$. This matrix $Q_{73}(v)$ is defined for all values of $\eta$. In the original paper [1] of 1972, the condition (4) is imposed and a matrix $Q(v)$ is constructed which is nondegenerate. We call this matrix $Q_{72}(v)$.

We have investigated this construction of $Q_{72}(v)$ in the series of papers [14–16]. In the first paper of the series [14], we found for $m_2 = 0$ and $m_1$ odd that the construction of [1] gives a $Q_{72}(v)$ matrix which commutes with $S$ but does not commute with $R$ and $RS$. We also found that the construction fails for $L$ odd and $m_1$ even. For this case a new construction was found in [16] which contains a parameter $t$. We call $Q_{72}^{(1)}(v)$ the $Q(v)$ matrix constructed by means of [1] and $Q_{72}^{(2)}(v; t)$ the matrix constructed by the use of the procedure of [16].

In this paper, we extend these constructions by considering the general case of (4) with $m_2 \neq 0$. We find that there are four distinct cases depending on the parity of $m_1$ and $m_2$. When necessary we denote these cases as $Q_{72oe}^{(1)}(v)$, $Q_{72oo}^{(1)}(v)$, $Q_{72eo}^{(1)}(v)$ and $Q_{72ee}^{(2)}(v; t)$, where the first (second) subscript indicates the parity of $m_1(m_2)$. These cases are distinguished by different commutation relations with the operators $S$, $R$ and $RS$ as follows.

*Case 1. $m_1$ odd, $m_2$ even, $N$ unrestricted*

$$\left[Q_{72oe}^{(1)}(v), S\right] = 0, \qquad \left[Q_{72oe}^{(1)}(v), R\right] \neq 0, \qquad \left[Q_{72oe}^{(1)}(v), RS\right] \neq 0. \tag{9}$$

*Case 2. $m_1$ odd, $m_2$ odd, $N$ unrestricted*

$$\left[Q_{72oo}^{(1)}(v), S\right] \neq 0, \qquad \left[Q_{72oo}^{(1)}(v), R\right] \neq 0, \qquad \left[Q_{72oo}^{(1)}(v), RS\right] = 0. \tag{10}$$

*Case 3. $m_1$ even, $m_2$ odd, $N$ unrestricted*

$$\left[Q_{72eo}^{(1)}(v), S\right] \neq 0, \qquad \left[Q_{72eo}^{(1)}(v), R\right] = 0, \qquad \left[Q_{72eo}^{(1)}(v), RS\right] \neq 0. \tag{11}$$

*Case 4.* $m_1$ even, $m_2$ even, $N$ even.

We find that there are matrices $Q_{72ee}^{(2)}(v; t)$ for $t = n\eta$ and $t = (n + 1/2)\eta$ with $n$ an integer, and thus there are several subcases to be distinguished:

*Case 4A.* $t = n\eta$

$$\left[ Q_{72ee}^{(2)}(v; n\eta), S \right] = 0, \qquad \left[ Q_{72ee}^{(2)}(v; n\eta), R \right] \neq 0, \qquad \left[ Q_{72ee}^{(2)}(v; n\eta), RS \right] \neq 0. \qquad (12)$$

*Case 4B.* $t = (n + 1/2)\eta$, $m_1 \equiv 0 (\mathrm{mod}\, 4)$ and $m_2 \equiv 0 (\mathrm{mod}\, 4)$

$$\begin{aligned}
\left[ Q_{72ee}^{(2)}(v; (n + 1/2)\eta), S \right] &= 0, \\
\left[ Q_{72ee}^{(2)}(v; (n + 1/2)\eta), R \right] &\neq 0, \\
\left[ Q_{72ee}^{(2)}(v; (n + 1/2)\eta), RS \right] &\neq 0.
\end{aligned} \qquad (13)$$

The matrix $Q_{72ee}^{(2)}(v; n\eta)$ is similar to the matrix $Q_{72ee}^{(2)}(v; (n' + 1/2)\eta)$.

*Case 4C.* $t = (n + 1/2)\eta$, $m_1 \equiv 2 (\mathrm{mod}\, 4)$, $m_2 \equiv 2 (\mathrm{mod}\, 4)$

$$\begin{aligned}
\left[ Q_{72ee}^{(2)}(v; (n + 1/2)\eta), S \right] &\neq 0, \\
\left[ Q_{72ee}^{(2)}(v; (n + 1/2)\eta), R \right] &= 0, \\
\left[ Q_{72ee}^{(2)}(v; (n + 1/2)\eta), RS \right] &\neq 0.
\end{aligned} \qquad (14)$$

*Case 4D.* $t = (n + 1/2)\eta$, $m_1 \equiv 0 (\mathrm{mod}\, 4)$, $m_2 \equiv 2 (\mathrm{mod}\, 4)$

$$\begin{aligned}
\left[ Q_{72ee}^{(2)}(v; (n + 1/2)\eta), S \right] &\neq 0, \\
\left[ Q_{72ee}^{(2)}(v; (n + 1/2)\eta), R \right] &\neq 0, \\
\left[ Q_{72ee}^{(2)}(v; (n + 1/2)\eta), RS \right] &= 0.
\end{aligned} \qquad (15)$$

No matrix $Q_{72ee}^{(2)}(v, (n + 1/2)\eta)$ exists for $m_1 \equiv 2 (\mathrm{mod}\, 4)$, $m_2 \equiv 0 (\mathrm{mod}\, 4)$. In addition, it is known from numerical computations [16] that for general values of $t$ a matrix can be constructed that satisfies (1) and (2) but not (3) which does not commute with any of $S$, $R$ or $RS$.

There remains one case for which no matrix has yet been found by the use of the methods of [1] or [16] which satisfies (1)–(3); this is $m_1$ even, $m_2$ even and $N$ odd. It has been seen in [15] from numerical computations that a $T Q$ equation for eigenvalues holds and the eigenvalues of $Q(v)$ have unique properties not seen in cases 1–4. The particular case $\eta = 2K/3$ is extensively treated in [10] where unique properties also exist for the six-vertex limit [17–20].

In section 2, we review the formalism of the eight-vertex model for the case that (4) holds with $m_2 \neq 0$. This involves a modification of the theta functions $\Theta(v)$ and $\mathrm{H}(v)$ which was first introduced in [2]. The properties of these modified theta functions are summarized in appendix A where we also prove various identities which will be used in the text. In section 3, we review the three steps of the construction of the $Q_{72}(v)$ matrix of [1] which uses the auxiliary matrices $Q_R(v)$ and $Q_L(v)$. The explicit construction of $Q_R(v)$ is given in section 4 with special attention to the recent discovery [16] that the principles of this construction lead to two different $Q$ matrices $Q_{72}^{(1)}(v)$ and $Q_{72}^{(2)}(v; t)$. In section 5, we use the methods of [1] to construct the $Q_{72}^{(1)}(v)$ which satisfies equations (1)–(3) for the three cases where $m_1$ and $m_2$ are not both even. In section 6, we consider case 4 when $N, m_1$ and $m_2$ are even and show that $Q_{72}^{(2)}(v; t)$ satisfies (1) with additional phase factors and that in cases 4A–4D the relations (2) and (3) are satisfied. The quasiperiodicity conditions and the general form of the eigenvalues of $Q_{72}^{(1)}(v)$, $Q_{72ee}^{(2)}(v; n\eta)$ and $Q_{72ee}^{(2)}(v; (n + 1/2)\eta)$ are derived in section 7 and we conclude in section 8 with a discussion of our results.

## 2. Formulation of the eight-vertex model

The Boltzmann weights of the eight-vertex model are given in terms of elements of a matrix $W_8(\alpha, \beta)_{\pm 1, \pm 1}$ in a two-dimensional space labeled by $\pm 1$ and an external two-dimensional space labeled by $\alpha = \pm 1, \beta = \pm 1$. These elements are given in terms of four quantities $a, b, c, d$ as

$$W_8(1, 1)|_{1,1} = W_8(-1, -1)|_{-1,-1} = a,$$
$$W_8(-1, -1)|_{1,1} = W_8(1, 1)|_{-1,-1} = b,$$

(16)

$$W_8(-1, 1)|_{1,-1} = W_8(1, -1)|_{-1,1} = c,$$
$$W_8(1, -1)|_{1,-1} = W_8(-1, 1)|_{-1,1} = d,$$

(17)

and the transfer matrix in the $2^N \times 2^N$ 'external space' is written as

$$T_8(v)|_{\alpha, \beta} = \mathrm{Tr} W_8(\alpha_1, \beta_1) W_8(\alpha_2, \beta_2) \cdots W_8(\alpha_N, \beta_N)$$

(18)

where the trace is in the 'internal' $2 \times 2$ space.

In the famous 1972 paper of Baxter [1], it was shown that any two transfer matrices commute if the four parameters for each of the two matrices are constrained by the two conditions

$$\frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)} = \Delta,$$

(19)

$$\frac{cd}{ab} = \Gamma.$$

(20)

These two homogeneous constraints on four parameters define a one-parameter family which satisfies

$$[T(v), T(v')] = 0.$$

(21)

The parameter $v$ is made explicit in the 1972 paper [1] by writing the Boltzmann weights in terms of the Jacobi elliptic functions

$$\mathrm{H}(v) = 2 \sum_{n=1}^{\infty} (-1)^{n-1} q^{(n-\frac{1}{2})^2} \sin[(2n - 1)\pi v/(2K)],$$

(22)

$$\Theta(v) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(nv\pi/K)$$
$$= -\mathrm{i} q^{1/4} \, \mathrm{e}^{\pi \, \mathrm{i} v/(2K)} \, \mathrm{H}(v + \mathrm{i} K'),$$

(23)

where $K$ and $K'$ are the standard elliptic integrals of the first kind and

$$q = \mathrm{e}^{-\pi K'/K}.$$

(24)

The parametrization of [1] is sufficient for the case $m_2 = 0$. However, to deal with the general case of (4) with $m_2 \neq 0$, Baxter in [2] introduces the 'modified' theta functions

$$\mathrm{H}_m(v) = \exp\left(\frac{\mathrm{i}\pi m_2}{8KL\eta}(v - K)^2\right) \mathrm{H}(v),$$
$$\Theta_m(v) = \exp\left(\frac{\mathrm{i}\pi m_2}{8KL\eta}(v - K)^2\right) \Theta(v).$$

(25)

In terms of $\Theta_m(v)$ and $H_m(v)$, the Boltzmann weights are parametrized as

$$
\begin{aligned}
a &= \Theta_m(-2\eta)\Theta_m(\eta - v)\, H_m(\eta + v),\\
b &= -\Theta_m(-2\eta)\, H_m(\eta - v)\Theta_m(\eta + v),\\
c &= -H_m(-2\eta)\Theta_m(\eta - v)\Theta_m(\eta + v),\\
d &= H_m(-2\eta)\, H_m(\eta - v)\, H_m(\eta + v).
\end{aligned}
\tag{26}
$$

When $m_2 = 0$, the parametrization (26) reduces to the parametrization of [1].

The factor in (25) is chosen so that the modified theta functions have the periodicity (see [2], equation (11))

$$
H_m(v + 4L\eta) = H_m(v), \qquad \Theta_m(v + 4L\eta) = \Theta_m(v).
\tag{27}
$$

In appendix A, we demonstrate that these modified theta functions are in fact Jacobi theta functions but that their fundamental parallelogram is no longer spanned by $2K$ and $2\mathrm{i}K'$ (the quasiperiods of $H(v)$ and $\Theta(v)$). Instead, we find the quasiperiodicity properties

$$
H_m(v + \omega_1) = (-1)^{r_1}(-1)^{r_1 r_2}\, H_m(v),
\tag{28}
$$

$$
\Theta_m(v + \omega_1) = (-1)^{r_1 r_2}\Theta_m(v),
\tag{29}
$$

$$
\begin{aligned}
H_m(v + \omega_2) &= (-1)^{b}(-1)^{ab}q'^{-1}\,\mathrm{e}^{-2\pi\,\mathrm{i}(v-K)/\omega_1}\, H_m(v)\\
&= (-1)^{a+b}(-1)^{ab}q'^{-1-r_2}\,\mathrm{e}^{-2\pi v/\omega_1}\, H_m(v),
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
\Theta_m(v + \omega_2) &= (-1)^{ab}q'^{-1}\,\mathrm{e}^{-2\pi\,\mathrm{i}(v-K/\omega_1)}\Theta_m(v)\\
&= (-1)^{a}(-1)^{ab}q'^{-1-r_2}\,\mathrm{e}^{-2\pi\,\mathrm{i}v/\omega_1}\Theta_m(v),
\end{aligned}
\tag{31}
$$

where the original (quasi)periods $2K, 2\mathrm{i}K'$ and the (quasi)periods $\omega_1, \omega_2$ are related by a modular transformation

$$
\omega_1 = 2(r_1 K + \mathrm{i}r_2 K'), \qquad \omega_2 = 2(bK + \mathrm{i}aK'),
\tag{32}
$$

$$
ar_1 - br_2 = 1.
\tag{33}
$$

Here, with $r_0$ defined as the greatest common divisor in $2m_1$ and $m_2$, the quantities $r_1$ and $r_2$ are given by

$$
2m_1 = r_0 r_1, \qquad m_2 = r_0 r_2.
\tag{34}
$$

From (33) the area of the fundamental period parallelogram

$$
0, \omega_1, \omega_1 + \omega_2, \omega_2
\tag{35}
$$

is $4KK'$. We thus see that the modified theta functions are in fact theta functions of nome

$$
q' = \mathrm{e}^{\mathrm{i}\pi\omega_2/\omega_1}
\tag{36}
$$

which are modular transforms of the original theta functions $\Theta(v)$ and $H(v)$. We also note that

$$
2L\eta = r_0\omega_1/2.
\tag{37}
$$

## 3. Formal construction of the matrices $Q_{72}(v)$

The construction of [1] of a matrix $Q$ which satisfies (1)–(3) under the condition (4) consists of three steps.

### 3.1. Construction of matrices $Q_R(v)$ and $Q_L(v)$

The first step begins with an assumption that there exists a matrix $Q_R(v)$ of the form

$$Q_R(v)|_{\alpha,\beta} = \text{Tr}\, S_R(\alpha_1, \beta_1) S_R(\alpha_2, \beta_2) \cdots S_R(\alpha_N, \beta_N), \tag{38}$$

with $S_R(\alpha, \beta)$ an $L \times L$ matrix with elements $s_{m,n}(\alpha, \beta)$ which satisfies

$$T(v) Q_R(v) = [h(v + \eta)]^N Q_R(v - 2\eta) + [h(v - \eta)]^N Q_R(v + 2\eta), \tag{39}$$

where

$$h(v) = \Theta_m(0)\Theta_m(-v)\, \text{H}_m(v). \tag{40}$$

This matrix $Q_R(v)$ cannot be unique because if (39) is multiplied on the left by any matrix $A$ which commutes with $T(v)$ then $A Q_R(v)$ also satisfies (39). In addition, we note that the matrix $\text{e}^{av} Q_R(v)$ will satisfy (38) with $h(v \pm \eta)^N$ replaced by $\text{e}^{\pm 2a\eta} h(v \pm \eta)^N$.

Similarly, we construct a matrix $Q_L(v)$

$$Q_L(v)|_{\alpha,\beta} = \text{Tr}\, S_L(\alpha_1, \beta_1) S_L(\alpha_2, \beta_2) \cdots S_L(\alpha_N, \beta_N) \tag{41}$$

which satisfies

$$Q_L(v) T(v) = [h(v + \eta)]^N Q_L(v - 2\eta) + [h(v - \eta)]^N Q_L(v + 2\eta). \tag{42}$$

This matrix is non-unique by multiplying on the right by any matrix which commutes with $T(v)$.

The matrices $Q_R(v)$ and $Q_L(v)$ are independently defined and can be independently constructed by analogous procedures. However, it is also instructive to note that the matrix $Q_L(v)$ can also be obtained by taking the transpose of (39) and using the symmetry properties of the transfer matrix

$$T^T(v) = (-1)^N T(-v), \tag{43}$$

$$T^T(v) = \text{e}^{\pi\, \text{i} N m_2(v-K)/L\eta} T(2K - v), \tag{44}$$

and the properties

$$h(v) = -h(-v), \tag{45}$$

$$h(v) = \text{e}^{\text{i}\pi m_2(v-K)/(L\eta)} h(2K - v), \tag{46}$$

we find constructions for $Q_L(v)$ as

$$Q_L(v) = Q_R^T(-v), \tag{47}$$

$$Q_L(v) = \text{e}^{\pi\, \text{i} m_2 v N/(2L\eta)} Q_R^T(2K - v), \tag{48}$$

where to obtain (47) we have used (43) and (45), and to obtain (48) we have used (44) and (46), and we note that (47) [(48)] may differ by right multiplication by a matrix $A$ which commutes with $T(v)$. The matrices $Q_R(v)$ and $Q_L(v)$ will not in general satisfy either (2) or (3).

### 3.2. The interchange relation

To satisfy conditions (2) and (3), we impose the interchange relation

$$Q_L(v_1) A Q_R(v_2) = Q_L(v_2) A Q_R(v_1), \tag{49}$$

where the matrix $A$ is independent of $v_1$ and $v_2$, satisfies $A^2 = 1$ and commutes with the transfer matrix $T(v)$. In this paper, we will consider the four choices

$$A = I, S, R, RS. \tag{50}$$

These choices may be thought of as representing the arbitrariness in the construction of $Q_R(v)$ and/or $Q_L(v)$. We will see below that for cases 1–3 that (49) holds for only two of the four choices of $A$, whereas for case 4 (49) holds for all four choices (50).

### 3.3. The nonsingularity condition

The final requirement is that the matrices $Q_R(v)$ and $Q_L(v)$ possess one value $v = v_0$ such that $Q_R(v_0)^{-1}$ and $Q_L(v_0)^{-1}$ exist. Under this nonsingularity assumption, we obtain from (39), (42) and (49) that the matrices

$$Q_{72}(v) = Q_R(v)Q_R^{-1}(v_0) = AQ_L^{-1}(v_0)Q_L(v)A \tag{51}$$

and

$$AQ_{72}(v)A = AQ_R(v)Q_R^{-1}(v_0)A = Q_L^{-1}(v_0)Q_L(v) \tag{52}$$

both satisfy the three conditions (1)–(3) needed for the *TQ* equation.

We will see below in cases 1–3 that $Q_R(v)$ is generically nonsingular but for case 4 where (49) holds for all four choices (50) that $Q_R(v)$ is singular for all $v$. In cases 1–3 where the interchange relation (49) holds for two and only two matrices $A_1$ and $A_2$, we obtain from (52)

$$A_1 Q_{72}(v)A_1 = A_2 Q_{72}(v)A_2 \tag{53}$$

or

$$A_2 A_1 Q_{72}(v)A_1 A_2 = Q_{72}(v). \tag{54}$$

If $A_1$, $A_2$ commute then $Q_{72}(v)$ commutes with $A_1 A_2$.

## 4. The matrices $Q_R(v)$ and $Q_L(v)$

In appendix C of [1], it is shown that for the existence of the matrix $Q_R(v)$ of the form (38) which satisfies (39) it is necessary that the matrix elements $s_{m,n}^R$ of $S_R$ satisfy

$$\begin{aligned}
(ap_n - bp_m)s_{m,n}^R(+, \beta) + (d - cp_m p_n)s_{m,n}^R(-, \beta) = 0, \\
(c - dp_m p_n)s_{m,n}^R(+\beta) + (bp_n - ap_m)s_{m,n}^R(-, \beta) = 0.
\end{aligned} \tag{55}$$

This set of homogeneous linear equations will have a nontrivial solution provided

$$(a^2 + b^2 - c^2 - d^2)p_m p_n = ab\left(p_m^2 + p_n^2\right) - cd\left(1 + p_m^2 p_n^2\right). \tag{56}$$

This can only happen for certain values of $m$ and $n$. For all other values, we have

$$s_{m,n}^R(\alpha, \beta) = 0. \tag{57}$$

Using the parametrizations (26), we have

$$\frac{a^2 + b^2 - c^2 - d^2}{ab} = 2\text{cn}(2\eta)\,\text{dn}(2\eta), \qquad cd/ab = k\,\text{sn}^2(2\eta), \tag{58}$$

where $\text{sn}(v)$, $\text{cn}(v)$, $\text{dn}(v)$ are the conventional doubly periodic functions with periods $2K$ and $2iK'$, and $\text{sn}(v)$ is given in terms of theta functions as

$$k^{1/2}\,\text{sn}(v) = \text{H}(v)/\Theta(v). \tag{59}$$

Thus, (56) becomes

$$2\text{cn}(2\eta)\,\text{dn}(2\eta)p_m p_n = p_m^2 + p_n^2 - k\,\text{sn}^2(2\eta)\left(1 + p_m^2 p_n^2\right) \tag{60}$$

and it is shown in [1] that if $p_m$ is written as

$$p_m = k^{1/2}\,\text{sn}(u) \tag{61}$$

it follows from (60) that

$$p_n = k^{1/2} \operatorname{sn}(u \pm 2\eta). \tag{62}$$

In order for the nonsingularity condition for $Q_R(v)$ to hold, we need additional nonvanishing elements $s_{m,n}^R$. We consider two possible choices.

### 4.1. The matrices $Q_R^{(1)}(v)$ and $Q_L^{(1)}(v)$

The first choice is to require

$$s_{1,1}(\alpha, \beta) \neq 0, \qquad s_{L,L}(\alpha, \beta) \neq 0. \tag{63}$$

Then, equation (60) has to be satisfied for $n = m$. Then,

$$\operatorname{sn}(u) = \operatorname{sn}(u \pm 2\eta). \tag{64}$$

This fixes the parameter $u$ to become $u = K \pm \eta$ and leads to the restriction to discrete $\eta$:

$$2L\eta = 2m_1 K + \mathrm{i} m_2 K'. \tag{65}$$

One obtains from (61) and (62) that

$$p_n = k^{1/2} \operatorname{sn}(K + (2n - 1)\eta). \tag{66}$$

We indicate the choice (63) by writing

$$S_R(\alpha, \beta) \rightarrow S_R^{(1)}(\alpha, \beta), \qquad Q_R(v) \rightarrow Q_R^{(1)}(v) \tag{67}$$

and from [1] we find

$$
\begin{aligned}
S_R^{(1)}(+, \beta)_{k,k+1}(v) &= \mathrm{H}_m(v + K - 2k\eta)\tau_{\beta,-k}, & 1 \leqslant k \leqslant L - 1, \\
S_R^{(1)}(+, \beta)_{k+1,k}(v) &= \mathrm{H}_m(v + K + 2k\eta)\tau_{\beta,k}, & 1 \leqslant k \leqslant L - 1, \\
S_R^{(1)}(+, \beta)_{1,1}(v) &= \mathrm{H}_m(v + K)\tau_{\beta,0}, & \\
S_R^{(1)}(+, \beta)_{L,L}(v) &= \mathrm{H}_m(v + K + 2L\eta)\tau_{\beta,L}, & \\
S_R^{(1)}(-, \beta)_{k,k+1}(v) &= \Theta_m(v + K - 2k\eta)\tau_{\beta,-k}, & 1 \leqslant k \leqslant L - 1, \\
S_R^{(1)}(-, \beta)_{k+1,k}(v) &= \Theta_m(v + K + 2k\eta)\tau_{\beta,k}, & 1 \leqslant k \leqslant L - 1, \\
S_R^{(1)}(-, \beta)_{1,1}(v) &= \Theta_m(v + K)\tau_{\beta,0}, & \\
S_R^{(1)}(-, \beta)_{L,L}(v) &= \Theta_m(v + K + 2L\eta)\tau_{\beta,L}. &
\end{aligned}
\tag{68}
$$

With this choice the argument of appendix C of [1] shows that equation (39) holds with $h(v)$ given by (40).

We choose to construct the matrix $Q_L(v)$ from (68) by use of (47) and (A.6) as

$$
\begin{aligned}
S_L^{(1)}(\alpha, +)_{k,k+1}(v) &= \mathrm{H}_m(v + K + 2k\eta)\tau'_{\alpha,-k}, & 1 \leqslant k \leqslant L - 1, \\
S_L^{(1)}(\alpha, +)_{k+1,k}(v) &= \mathrm{H}_m(v + K - 2k\eta)\tau'_{\alpha,k}, & 1 \leqslant k \leqslant L - 1, \\
S_L^{(1)}(\alpha, +)_{1,1}(v) &= \mathrm{H}_m(v + K)\tau'_{\alpha,0}, & \\
S_L^{(1)}(\alpha, +)_{L,L}(v) &= \mathrm{H}_m(v + K - 2L\eta)\tau'_{\alpha,L}, & \\
S_L^{(1)}(\alpha, -)_{k,k+1}(v) &= \Theta_m(v + K + 2k\eta)\tau'_{\alpha,-k}, & 1 \leqslant k \leqslant L - 1, \\
S_L^{(1)}(\alpha, -)_{k+1,k}(v) &= \Theta_m(v + K - 2k\eta)\tau'_{\alpha,k}, & 1 \leqslant k \leqslant L - 1, \\
S_L^{(1)}(\alpha, -)_{1,1}(v) &= \Theta_m(v + K)\tau'_{\alpha,0}, & \\
S_L^{(1)}(\alpha, -)_{L,L}(v) &= \Theta_m(v + K - 2L\eta)\tau'_{\alpha,L}. &
\end{aligned}
\tag{69}
$$

In order to cover the case $m_2 \neq 0$ we have to define $S_R$ and $S_L$ in terms of modified theta functions. That this is allowed is immediately obvious from the simple observation that

$$k^{1/2} \, \text{sn}(v) = \text{H}(v)/\Theta(v) = \text{H}_m(v)/\Theta_m(v). \tag{70}$$

## 4.2. The matrices $Q_R^{(2)}(v; t)$ and $Q_L^{(2)}(v; t)$ for N even

The second choice which exists for $m_1$ and $m_2$ even was recently found in [16] with

$$s_{1,L}^R(\alpha, \beta) \neq 0, \qquad s_{L,1}^R(\alpha, \beta) \neq 0. \tag{71}$$

This choice will always give a vanishing matrix $Q_R^{(2)}$ when used in (38) when $N$ is odd. Consequently, whenever we consider $Q_R^{(2)}(v; t)$ we will always assume that $N$ is even.

To obtain this case we need to have (60) hold for $m = 1, n = L$ and $m = L, n = 1$ which because of the symmetry in (60) in $m$ and $n$ gives the single equation

$$\text{sn}^2(v_r + 2\eta) + \text{sn}^2(v_r + 2L\eta) - \text{sn}^2 2\eta(1 + k^2 \text{sn}^2(v_r + 2\eta) \, \text{sn}^2(v_r + 2L\eta))$$
$$- 2\text{sn}(v_r + 2\eta) \, \text{sn}(v_r + 2L\eta) \, \text{cn} 2\eta \, \text{dn} 2\eta = 0. \tag{72}$$

This equation will hold if $p_n$ is given by (62) with $p_1 = p_{L+1}$ and thus

$$\text{sn}(v_r + 2\eta) = \text{sn}(v_r + 2(L+1)\eta) \tag{73}$$

which, using the periodicity properties $\text{sn}(v + 2K) = -\text{sn} \, v$ and $\text{sn}(v + 2iK') = \text{sn} \, v$, is satisfied for all $v$ if

$$2L\eta = 4\tilde{m}_1 K + 2i\tilde{m}_2 K' \tag{74}$$

which is the root of unity condition (4) with $m_1 = 2\tilde{m}_1, m_2 = 2\tilde{m}_2$. In other words, we are restricted to $m_1$ and $m_2$ even in the root of unity condition (4).

We will follow the notation of [16] by setting

$$v_r = t - \eta. \tag{75}$$

Then, indicating the choice (71) by writing

$$S_R(\alpha, \beta) \rightarrow S_R^{(2)}(\alpha, \beta), \qquad Q_R(v) \rightarrow Q_R^{(2)}(v), \tag{76}$$

we have

$$p_n = k^{1/2} \, \text{sn}[t + (2n-1)\eta] = \text{H}_m(t + (2n-1)\eta)/\Theta_m(t + (2n-1)\eta), \tag{77}$$

and following [16] we find

$$\begin{aligned}
S_R^{(2)}(+, \beta)_{k,k+1}(v) &= -\text{H}_m(v - t - 2k\eta)\tau_{\beta,-k}, \\
S_R^{(2)}(+, \beta)_{k+1,k}(v) &= \text{H}_m(v + t + 2k\eta)\tau_{\beta,k}, \\
S_R^{(2)}(-, \beta)_{k,k+1}(v) &= \Theta_m(v - t - 2k\eta)\tau_{\beta,-k}, \\
S_R^{(2)}(-, \beta)_{k+1,k}(v) &= \Theta_m(v + t + 2k\eta)\tau_{\beta,k}, \\
S_R^{(2)}(+, \beta)_{1,L}(v) &= \text{H}_m(v + t + 2L\eta)\tau_{\beta,L}, \\
S_R^{(2)}(+, \beta)_{L,1}(v) &= -\text{H}_m(v - t - 2L\eta)\tau_{\beta,-L}, \\
S_R^{(2)}(-, \beta)_{1,L}(v) &= \Theta_m(v + t + 2L\eta)\tau_{\beta,L}, \\
S_R^{(2)}(-, \beta)_{L,1}(v) &= \Theta_m(v - t - 2L\eta)\tau_{\beta,-L}.
\end{aligned} \tag{78}$$

With the choice (78) for $S_R^{(2)}$, we may follow the procedure of [1] to obtain the slight generalization of (1)

$$T(v)Q_R^{(2)}(v; t) = \omega^{-N}[h(v + \eta)]^N Q_R^{(2)}(v - 2\eta; t) + \omega^N [h(v - \eta)]^N Q_R^{(2)}(v + 2\eta; t), \tag{79}$$

where

$$\omega = \exp\left(\frac{i\pi m_2}{2L}\right).\tag{80}$$

The details of this computation which show the origin of the phase factor $\omega$ are given in appendix C.

The companion matrix $Q_L^{(2)}(v; t)$ must satisfy

$$Q_L^{(2)}(v, t)T(v) = \omega^{-N}[h(v + \eta)]^N Q_L^{(2)}(v - 2\eta; t) + \omega^N[h(v - \eta)]^N Q_L^{(2)}(v + 2\eta; t).\tag{81}$$

We find it convenient to use (43) and (45) to construct $Q_L^{(2)}(v; t)$ in terms of $Q_R^{(2)}(v; t)$ as

$$Q_L^{(2)}(v; t) = -Q_R^{(2)T}(2K - v; t)S,\tag{82}$$

where the factor of $-S$ which is inserted for convenience uses the non-uniqueness of $Q_L(v)$ under multiplication on the right by any matrix which commutes with $T(v)$. Thus, we find

$$\begin{aligned}
S_L^{(2)}(\alpha, +)_{k,k+1}(v) &= \mathrm{H}_m(v + t + 2k\eta)\tau'_{\alpha,-k},\\
S_L^{(2)}(\alpha, +)_{k+1,k}(v) &= -\mathrm{H}_m(v - t - 2k\eta)\tau'_{\alpha,k},\\
S_L^{(2)}(\alpha, -)_{k,k+1}(v) &= \Theta_m(v + t + 2k\eta)\tau'_{\alpha,-k},\\
S_L^{(2)}(\alpha, -)_{k+1,k}(v) &= \Theta_m(v - t - 2k\eta)\tau'_{\alpha,k},\\
S_L^{(2)}(\alpha, +)_{1,L}(v) &= -\mathrm{H}_m(v - t - 2L\eta)\tau'_{\alpha,L},\\
S_L^{(2)}(\alpha, +)_{L,1}(v) &= \mathrm{H}_m(v + t + 2L\eta)\tau'_{\alpha,-L},\\
S_L^{(2)}(\alpha, -)_{1,L}(v) &= \Theta_m(v - t - 2L\eta)\tau'_{\alpha,L},\\
S_L^{(2)}(\alpha, -)_{L,1}(v) &= \Theta_m(v + t + 2L\eta)\tau'_{\alpha,-L}.
\end{aligned}\tag{83}$$

## 5. The matrices $Q_{72}^{(1)}(v)$ for $m_1$ and $m_2$ not both even

The construction of the matrices $Q_R^{(1)}(v)$ and $Q_L^{(1)}(v)$ given in the previous section is valid for all integers $m_1$ and $m_2$ in the root of unity condition (4). However, the validity and the choice of the matrix $A$ in the interchange relation (49) and the nonsingularity condition are different for the different parities of $m_1$ and $m_2$.

### 5.1. The interchange relations

The computation of the interchange relations (49) is similar for all four choices of the matrix $A$ but each case differs in details. Therefore, we will treat the four cases separately. The results are summarized in section 5.1.5.

### 5.1.1. The case $A = I$.

We first consider the interchange relation (49) with $A = 1$ and write

$$Q_L^{(1)}(v')Q_R^{(1)}(v)|_{\alpha,\beta} = \mathrm{Tr}\, W^{(1)}(\alpha_1, \beta_1|v', v) \cdots W^{(1)}(\alpha_N, \beta_N|v', v),\tag{84}$$

where $W^{(1)}(\alpha, \beta|v', v)$ are $L^2 \times L^2$ matrices with elements

$$W^{(1)}(\alpha, \beta|v', v)_{k,k';l,l'} = \sum_{\gamma=\pm} S_L^{(1)}(\alpha, \gamma|v')_{k,l} S_R^{(1)}(\gamma, \beta|v)_{k',l'}.\tag{85}$$

Thus, the interchange relation (49) with $A = I$ will follow if we can show that there exists an $L^2 \times L^2$ diagonal matrix $Y$ with elements

$$y_{k,k';l,l'}^{(1)} = y_{k,k'}^{(1)}\delta_{k,l}\delta_{k',k'}\tag{86}$$

such that

$$W^{(1)}(\alpha, \beta|v', v) = Y^{(1)} W^{(1)}(\alpha, \beta|v, v') Y^{(1)-1}. \tag{87}$$

To examine the possibility of the existence of such a diagonal similarity transformation, we need to explicitly compute $W(\alpha, \beta|v', v)$ from (85). To do this, we use the identity

$$\Theta_m(v') \Theta_m(v) + H_m(v') H_m(v) = f_+(v + v') g_+(v' - v), \tag{88}$$

with

$$f_+(z) = -\frac{2q^{1/4}}{H(K)\Theta(K)} \exp\left(\frac{i\pi m_2}{8KL\eta}(K'^2 + 2iKK' - 2Kz)\right)$$
$$\times H_m((iK' + z)/2) H_m((iK' - z)/2), \tag{89}$$

$$g_+(z) = H_m((iK' + z)/2 + K) H_m((iK' - z)/2 + K), \tag{90}$$

where we note the following properties:

$$f_+(-z) = e^{i\pi m_2 z/(2L\eta)} f_+(z), \tag{91}$$

$$g_+(-z) = g_+(z), \tag{92}$$

$$g_+(z + 4L\eta) = (-1)^{m_1 m_2} g_+(z), \tag{93}$$

and for $m_1$ and $m_2$ both even

$$g_+(v + 2L\eta) = (-1)^{m_1 m_2/4} g_+(z). \tag{94}$$

The properties (91) and (92) are obvious from the definitions (89) and (90). Property (94) follows from (93) with $m_1 \to m_1/2$ and $m_2 \to m_2/2$. The relation (93) follows from (A.7)–(A.10). Using (88) we find explicitly

$W^{(1)}(\alpha, \beta|v', v)_{k,k',l,l'}$

$$\begin{aligned}
&= \delta_{k+1,l}\delta_{k'+1,l'}\tau'_{\alpha,-k}\tau_{\beta,-k'} f_+(v' + v + 2K + 2(k - k')\eta)g_+(v' - v + 2(k + k')\eta) \\
&+ \delta_{k+1,l}\delta_{k',l'+1}\tau'_{\alpha,-k}\tau_{\beta,l'} f_+(v' + v + 2K + 2(k + l')\eta)g_+(v' - v + 2(k - l')\eta) \\
&+ \delta_{k+1,l}\delta_{k',1}\delta_{l',1}\tau'_{\alpha,-k}\tau_{\beta,0} f_+(v' + v + 2K + 2k\eta)g_+(v' - v + 2k\eta) \\
&+ \delta_{k+1,l}\delta_{k',L}\delta_{l',L}\tau'_{\alpha,-k}\tau_{\beta,L} f_+(v' + v + 2K + 2(k + L)\eta)g_+(v' - v + 2(k - L)\eta) \\
&+ \delta_{k,l+1}\delta_{k'+1,l'}\tau'_{\alpha,l}\tau_{\beta,-k'} f_+(v' + v + 2K - 2(l + k')\eta)g_+(v' - v - 2(l - k')\eta) \\
&+ \delta_{k,l+1}\delta_{k',l'+1}\tau'_{\alpha,l}\tau_{\beta,l'} f_+(v' + v + 2K - 2(l - l')\eta)g_+(v' - v - 2(l + l')\eta) \\
&+ \delta_{k,l+1}\delta_{k',1}\delta_{l',1}\tau'_{\alpha,l}\tau_{\beta,0} f_+(v' + v + 2K - 2l\eta)g_+(v' - v - 2l\eta) \\
&+ \delta_{k,l+1}\delta_{k',L}\delta_{l',L}\tau'_{\alpha,l}\tau_{\beta,L} f_+(v' + v + 2K - 2(l - L)\eta)g_+(v' - v - 2(l + L)\eta) \\
&+ \delta_{k,1}\delta_{l,1}\delta_{k'+1,l'}\tau'_{\alpha,0}\tau_{\beta,-k'} f_+(v' + v + 2K - 2k'\eta)g_+(v' - v + 2k'\eta) \\
&+ \delta_{k,1}\delta_{l,1}\delta_{k',l'+1}\tau'_{\alpha,0}\tau_{\beta,l'} f_+(v' + v + 2K + 2l'\eta)g_+(v' - v - 2l'\eta) \\
&+ \delta_{k,L}\delta_{l,L}\delta_{k'+1,l'}\tau'_{\alpha,L}\tau_{\beta,-k'} f_+(v' + v + 2K - 2(L + k')\eta)g_+(v' - v - 2(L - k')\eta) \\
&+ \delta_{k,L}\delta_{l,L}\delta_{k',l'+1}\tau'_{\alpha,L}\tau_{\beta,l'} f_+(v' + v + 2K - 2(L - l')\eta)g_+(v' - v - 2(L + l')\eta) \\
&+ \delta_{k,1}\delta_{l,1}\delta_{k',1}\delta_{l',1}\tau'_{\alpha,0}\tau_{\beta,0} f_+(v' + v + 2K)g_+(v' - v) \\
&+ \delta_{k,1}\delta_{l,1}\delta_{k',L}\delta_{l',L}\tau'_{\alpha,0}\tau_{\beta,L} f_+(v' + v + 2K + 2L\eta)g_+(v' - v - 2L\eta) \\
&+ \delta_{k,L}\delta_{l,L}\delta_{k',1}\delta_{l',1}\tau'_{\alpha,L}\tau_{\beta,0} f_+(v' + v + 2K - 2L\eta)g_+(v' - v - 2L\eta) \\
&+ \delta_{k,L}\delta_{l,L}\delta_{k',L}\delta_{l',L}\tau'_{\alpha,L}\tau_{\beta,L} f_+(v' + v + 2K)g_+(v' - v - 4L\eta). \tag{95}
\end{aligned}$$

A necessary condition for the existence of a diagonal similarity transformation is that the diagonal elements $W^{(1)}(\alpha, \beta|v', v)_{k,k';k,k'}$ and $W^{(1)}(\alpha, \beta|v, v')_{k,k';k,k'}$ be equal. From the last four terms in (95) we find that these diagonal elements are

$$W^{(1)}(\alpha, \beta | v', v)_{11,11} = f_+(v' + v + 2K)g_+(v' - v)\tau'_{\alpha,0}\tau_{\beta,0}, \tag{96}$$

$$W^{(1)}(\alpha, \beta | v', v)_{1,L;1,L} = f_+(v' + v + 2K + 2L\eta)g_+(v' - v - 2L\eta)\tau'_{\alpha,0}\tau_{\beta,L}, \tag{97}$$

$$W^{(1)}(\alpha, \beta | v', v)_{L,1;L,1} = f_+(v' + v + 2K - 2L\eta)g_+(v' - v - 2L\eta)\tau'_{\alpha,L}\tau_{\beta,0}, \tag{98}$$

$$W(\alpha, \beta | v', v)_{L,L;L,L} = f_+(v' + v + 2K)g_+(v' - v - 4L\eta)\tau'_{\alpha,L}\tau_{\beta,L}. \tag{99}$$

It follows from (96) and (99) by use of (93) that

$$W^{(1)}(\alpha, \beta | v', v)_{1,1;1,1} = W^{(1)}(\alpha, \beta | v, v')_{1,1;1,1}, \tag{100}$$

$$W^{(1)}(\alpha, \beta | v', v)_{L,L;L,L} = W^{(1)}(\alpha, \beta | v, v')_{L,L;L,L}. \tag{101}$$

To examine the elements $W^{(1)}(\alpha, \beta | v', v)_{1,L;1,L}$ and $W^{(1)}(\alpha, \beta | v', v)_{L,1;L,1}$ we use the identity (93) in (97) and (98). We find

$$W^{(1)}(\alpha, \beta | v', v)_{1,L;1,L} = (-1)^{m_1 m_2} W^{(1)}(\alpha, \beta | v, v')_{1,L;1,L}, \tag{102}$$

$$W^{(1)}(\alpha, \beta | v', v)_{L,1;L,1} = (-1)^{m_1 m_2} W^{(1)}(\alpha, \beta | v, v')_{L,1;L,1}. \tag{103}$$

We thus conclude that the interchange relation (49) is not satisfied if $m_1$ and $m_2$ are both odd. However, when at least one of the integers $m_1, m_2$ is even we do have the necessary equality and the remainder of the proof of the existence of the diagonal matrix $Y$ as given in appendix C of [1] holds. We thus conclude that in the case that when $m_1 m_2$ is even that (49) holds.

*5.1.2. The case $A = S$.* The proof of the interchange relation (49) with $A = S$ is similar to the proof with $A = I$.

We first write

$$Q_L^{(1)}(v')SQ_R^{(1)}(v)|_{\alpha,\beta} = \text{Tr}\, W^{(1)S}(\alpha_1, \beta_1 | v', v) \cdots W^{(1)S}(\alpha_N, \beta_N | v', v), \tag{104}$$

where $W^{(1)S}(\alpha, \beta | v', v)$ are $L^2 \times L^2$ matrices with elements

$$W^{(1)S}(\alpha, \beta | v', v)_{k,k';l,l'} = \sum_{\gamma=\pm} \gamma\, S_L(\alpha, \gamma | v')_{k,l} S_R(\gamma, \beta | v)_{k',l'}. \tag{105}$$

Thus, (49) will follow if we can show that there exists an $L^2 \times L^2$ diagonal matrix $Y^S$ with elements

$$y_{k,k';l,l'}^{(1)S} = y_{k,k'}^{(1)S} \delta_{k,l} \delta_{k',l'} \tag{106}$$

such that

$$W^{(1)S}(\alpha, \beta | v', v) = Y^{(1)S} W^S(\alpha, \beta | v, v') Y^{(1)S-1}. \tag{107}$$

To explicitly compute the matrix $W^{(1)S}(\alpha, \beta | v', v)$, we use the identity which follows immediately from identities (88) and (A.7) by sending $v' \to -v'$

$$H_m(u)\, H_m(v) - \Theta_m(u)\Theta_m(v) = f_-(u + v)g_-(u - v), \tag{108}$$

where

$$f_-(z) = \frac{2q^{1/4}}{H(K)\Theta(K)} \exp\left(\frac{i\pi m_2}{8KL\eta}(K'^2 + 2iKK' - 2Kz)\right)$$
$$\times H_m((iK' + z)/2 + K)\, H_m((iK' - z)/2 + K), \tag{109}$$

$$g_-(z) = H_m((iK' + z)/2)H_m((iK' - z)/2), \tag{110}$$

which have the properties that

$$f_-(-z) = e^{i\pi m_2 z/(2L\eta)} f_-(z), \tag{111}$$

$$g_-(-z) = g_-(z), \tag{112}$$

$$g_-(z + 4L\eta) = (-1)^{m_1 m_2}(-1)^{m_2} g_-(z), \tag{113}$$

and for $m_1$ and $m_2$ both even

$$g_-(z + 2L\eta) = (-1)^{m_1 m_2/4}(-1)^{m_2/2} g_-(z). \tag{114}$$

The property (114) follows from (113) with $m_1 \to m_1/2$ and $m_2 \to m_2/2$. The proof of (113) follows from (A.7)–(A.10). The properties (111) and (112) are obvious from the definitions (109) and (110):

$$
\begin{aligned}
W^{(1)S}&(\alpha, \beta|v', v)_{k,k';l,l'} \\
&= + \delta_{k+1,l}\delta_{k'+1,l'}\tau'_{\alpha,-k}\tau_{\beta,-k'} f_-(v' + v + 2K + 2(k - k')\eta)g_-(v' - v + 2(k + k')\eta) \\
&\quad + \delta_{k+1,l}\delta_{k',l'+1}\tau'_{\alpha,-k}\tau_{\beta,l'} f_-(v' + v + 2K + 2(k + l')\eta)g_-(v' - v + 2(k - l')\eta) \\
&\quad + \delta_{k+1,l}\delta_{k',1'}\delta_{l',1}\tau'_{\alpha,-k}\tau_{\beta,0} f_-(v' + v + 2K + 2k\eta)g_-(v' - v + 2k\eta) \\
&\quad + \delta_{k+1,l}\delta_{k',L}\delta_{l',L}\tau'_{\alpha,-k}\tau_{\beta,L} f_-(v' + v + 2K + 2(k + L)\eta)g_-(v' - v + 2(k - L)\eta) \\
&\quad + \delta_{k,l+1}\delta_{k'+1,l'}\tau'_{\alpha,l}\tau_{\beta,-k'} f_-(v' + v + 2K - 2(l + k')\eta)g_-(v' - v - 2(l - k')\eta) \\
&\quad + \delta_{k,l+1}\delta_{k',l'+1}\tau'_{\alpha,l}\tau_{\beta,l'} f_-(v' + v + 2K - 2(l - l')\eta)g_-(v' - v - 2(l + l')\eta) \\
&\quad + \delta_{k,l+1}\delta_{k',1'}\delta_{l',1}\tau'_{\alpha,l}\tau_{\beta,0} f_-(v' + v + 2K - 2l\eta)g_-(v' - v - 2l\eta) \\
&\quad + \delta_{k,l+1}\delta_{k',L}\delta_{l',L}\tau'_{\alpha,l}\tau_{\beta,L} f_-(v' + v + 2K - 2(l - L)\eta)g_-(v' - v - 2(l + L)\eta) \\
&\quad + \delta_{k,1}\delta_{l,1}\delta_{k'+1,l'}\tau'_{\alpha,0}\tau_{\beta,-k'} f_-(v' + v + 2K - 2k'\eta)g_-(v' - v + 2k'\eta) \\
&\quad + \delta_{k,1}\delta_{l,1}\delta_{k',l'+1}\tau'_{\alpha,0}\tau_{\beta,l'} f_-(v' + v + 2K + 2l'\eta)g_-(v'-v-2l'\eta) \\
&\quad + \delta_{k,L}\delta_{l,L}\delta_{k'+1,l'}\tau'_{\alpha,L}\tau_{\beta,-k'} f_-(v' + v + 2K - 2(L + k')\eta)g_-(v'-v-2(L-k')\eta) \\
&\quad + \delta_{k,L}\delta_{l,L}\delta_{k',l'+1}\tau'_{\alpha,L}\tau_{\beta,l'} f_-(v' + v + 2K - 2(L - l')\eta)g_-(v' - v - 2(L + l')\eta) \\
&\quad + \delta_{k,1}\delta_{l,1}\delta_{k',1'}\delta_{l',1}\tau'_{\alpha,0}\tau_{\beta,0} f_-(v' + v + 2K)g_-(v' - v) \\
&\quad + \delta_{k,1}\delta_{l,1}\delta_{k',L}\delta_{l',L}\tau'_{\alpha,0}\tau_{\beta,L} f_-(v' + v + 2K + 2L\eta)g_-(v' - v - 2L\eta) \\
&\quad + \delta_{k,L}\delta_{l,L}\delta_{k',1'}\delta_{l',1}\tau'_{\alpha,L}\tau_{\beta,0} f_-(v' + v + 2K - 2L\eta)g_-(v' - v - 2L\eta) \\
&\quad + \delta_{k,L}\delta_{l,L}\delta_{k',L}\delta_{l',L}\tau'_{\alpha,L}\tau_{\beta,L} f_-(v' + v + 2K)g_-(v' - v - 4L\eta).
\end{aligned}
$$

In order for the diagonal matrix $Y^{(1)S}$ to exist, it is necessary that the diagonal elements of $W^{(1)S}(\alpha, \beta|v', v)_{k,k';k,k'}$ be symmetric under the interchange of $v'$ and $v$. From the last four terms in (115), these diagonal elements are

$$W^{(1)S}(\alpha, \beta|v', v)_{1,1;1,1} = f_-(v' + v + 2K)g_-(v' - v)\tau'_{\alpha,0}\tau_{\beta,0}, \tag{115}$$

$$W^{(1)S}(\alpha, \beta|v', v)_{1,L;1,L} = f_-(v' + v + 2K + 2L\eta)g_-(v' - v - 2L\eta)\tau'_{\alpha,0}\tau_{\beta,L}, \tag{116}$$

$$W^{(1)S}(\alpha, \beta|v', v)_{L,1;L,1} = f_-(v' + v + 2K - 2L\eta)g_-(v' - v - 2L\eta)\tau'_{\alpha,L}\tau_{\beta,0}, \tag{117}$$

$$W^{(1)S}(\alpha, \beta|v', v)_{L,L;L,L} = f_-(v' + v + 2K)g_-(v' - v - 4L\eta)\tau'_{\alpha,L}\tau_{\beta,L}. \tag{118}$$

The equalities

$$W^{(1)S}(\alpha, \beta|v', v)_{1,1;1,1} = W^{(1)S}(\alpha, \beta|v, v')_{1,1;1,1}, \tag{119}$$

$$W^{(1)S}(\alpha, \beta|v', v)_{L,L;L,L} = W^{(1)S}(\alpha, \beta|v, v')_{L,L;L,L} \tag{120}$$

follow from (113).

To study $W^{(1)S}(\alpha, \beta|v', v)_{1,L;1,L}$ and $W^{(1)S}(\alpha, \beta|v', v)_{L,1;L,1}$, we use the identity (113) in (116) and (117) to obtain

$$W^{(1)S}(\alpha, \beta|v', v)_{1,L;1,L} = (-1)^{m_1 m_2}(-1)^{m_2} W^{(1)S}(\alpha, \beta|v, v')_{1,L;1,L}, \tag{121}$$

$$W^{(1)S}(\alpha, \beta|v', v)_{L,1;L,1} = (-1)^{m_1 m_2}(-1)^{m_2} W^{(1)S}(\alpha, \beta|v, v')_{L,1;L,1}. \tag{122}$$

From (121) and (122) we conclude that in order for (49) to hold with $A = S$ both $m_1$ and $m_2$ must be odd or $m_2$ must be even. With this restriction, the method of appendix C of [1] demonstrates the existence of the similarity transformation $Y$ and thus (49) with $A = S$ is proven.

*5.1.3. The case $A = R$.* For the case $A = R$, we consider

$$Q_L^{(1)}(v') R Q_R^{(1)}(v)|_{\alpha, \beta} = \text{Tr} \, W^{(1)R}(\alpha_1, \beta_1|v', v) \cdots W^{(1)R}(\alpha_N, \beta_N|v', v), \tag{123}$$

where $W^{(1)R}(\alpha, \beta|v', v)$ are $L^2 \times L^2$ matrices with elements

$$W^{(1)R}(\alpha, \beta|v', v)_{k,k';l,l'} = \sum_{\gamma=\pm} S_L^{(1)}(\alpha, \gamma|v')_{k,l} S_R^{(1)}(-\gamma, \beta|v)_{k',l'} \tag{124}$$

and use the identity derived from 15.4.28 of [21]

$$\Theta_m(v_1) \, H_m(v_2) + H_m(v_1) \Theta_m(v_2) = f_+^R(v_1 + v_2) g_+^R(v_1 - v_2), \tag{125}$$

with

$$f_+^R(z) = 2H_m(z/2)\Theta_m(z/2)/(H_m(K)\Theta_m(K)), \tag{126}$$

$$g_+^R(z) = H_m(K + z/2)\Theta_m(K + z/2), \tag{127}$$

where we note that

$$f_+^R(-z) = -e^{i\pi m_2 z/(2L\eta)} f_+^R(z), \tag{128}$$

$$g_+^R(-z) = g_+^R(z), \tag{129}$$

$$g_+^R(z + 4L\eta) = (-1)^{m_1}(-1)^{m_1 m_2} g_+^R(z) \tag{130}$$

and for $m_1$ and $m_2$ even

$$g_+^R(z + 2L\eta) = (-1)^{m_1/2}(-1)^{m_1 m_2/4} g_+^R(z), \tag{131}$$

where (129) follows from (127). The proof of (130) follows from (A.7)–(A.10). The property (131) follows from (130) with $m_1 \to m_1/2$ and $m_2 \to m_2/2$.

The diagonal elements of $W^{(1)R}(\alpha, \beta|v', v)$ are

$$W^{(1)R}(\alpha, \beta|v', v)_{11,11} = f_+^R(v + v' + 2k) g_+^R(v - v') \tau'_{\alpha,0} \tau_{\beta,0}, \tag{132}$$

$$W^{(1)R}(\alpha, \beta|v', v)_{1,L;1,L} = f_+^R(v + v' + 2K + 2L\eta) g_+^R(v - v' + 2L\eta) \tau'_{\alpha,0} \tau_{\beta,L}, \tag{133}$$

$$W^{(1)R}(\alpha, \beta|v', v)_{L,1;L,1} = f_+^R(v + v' + 2K - 2L\eta) g_+^R(v - v' + 2L\eta) \tau'_{\alpha,L} \tau_{\beta,0}, \tag{134}$$

$$W^{(1)R}(\alpha, \beta|v', v)_{L,L;L,L} = f_+^R(v + v' + 2K) g_+^R(v - v' + 4L\eta) \tau'_{\alpha,L} \tau_{\beta,L}. \tag{135}$$

The equalities

$$W^{(1)R}(\alpha, \beta|v', v)_{1,1;1,1} = W^{(1)R}(\alpha, \beta|v, v')_{1,1;1,1}, \tag{136}$$

$$W^{(1)R}(\alpha, \beta|v', v)_{L,L;L,L} = W^{(1)R}(\alpha, \beta|v, v')_{L,L;L,L} \tag{137}$$

follow from (130), and

$$W^{(1)R}(\alpha, \beta|v', v)_{1,L;1,L} = (-1)^{m_1 m_2}(-1)^{m_1} W^{(1)R}(\alpha, \beta|v, v')_{1,L;1,L}, \tag{138}$$

$$W^{(1)R}(\alpha, \beta|v', v)_{L,1;L,1} = (-1)^{m_1 m_2}(-1)^{m_1} W^{(1)R}(\alpha, \beta|v, v')_{L,1;L,1} \tag{139}$$

follow from (130). Consequently, with the restriction that $m_1$ is even or that both $m_1$ and $m_2$ are odd the methods of appendix C of [1] demonstrate that the interchange relation (49) holds for $A = R$.

*5.1.4. The case $A = RS$.* For the case $A = RS$, we consider

$$Q_L^{(1)}(v')RSQ_R^{(1)}(v)|_{\alpha,\beta} = \text{Tr } W^{(1)RS}(\alpha_1, \beta_1|v', v) \cdots W^{(1)RS}(\alpha_N, \beta_N|v', v), \tag{140}$$

where $W^{(1)RS}(\alpha, \beta|v', v)$ are $L^2 \times L^2$ matrices with elements

$$W^{(1)RS}(\alpha, \beta|v', v)_{k,k';l,l'} = \sum_{\gamma=\pm} \gamma S_L^{(1)}(\alpha, \gamma|v')_{k,l} S_R^{(1)}(-\gamma, \beta|v)_{k',l'} \tag{141}$$

and use the identity

$$\Theta_m(v_1) H_m(v_2) - H_m(v_1)\Theta_m(v_2) = f_-^R(v_1 + v_2)g_-^R(v_1 - v_2)) \tag{142}$$

with

$$f_-^R(z) = 2 e^{-i\pi m_2 K/2L\eta} H_m(z/2 - K)\Theta_m(-z/2 + K)/(H_m(K)\Theta_m(K), \tag{143}$$

$$g_-^R(z) = H_m(z/2)\Theta_m(-z/2), \tag{144}$$

where we note

$$g_-^R(-z) = -g_-^R(z), \tag{145}$$

$$g_-^R(z + 4L\eta) = (-1)^{m_1+m_2}(-1)^{m_1 m_2} g_-^R(z), \tag{146}$$

and for both $m_1$ and $m_2$ even

$$g_-^R(z + 2L\eta) = (-1)^{(m_1+m_2)/2}(-1)^{m_1 m_2/4} g_-^R(z). \tag{147}$$

The relation (145) follows from (A.7), the relation (147) follows from (146) with $m_1 \to m_1/2$, $m_2 \to m_2/2$, and the proof of (146) follows from (A.7)–(A.10). The diagonal elements of $W^{(1)R}(\alpha, \beta|v', v)$ are

$$W^{(1)RS}(\alpha, \beta|v', v)_{11,11} = f_-^R(v + v' + 2K)g_-^R(v - v')\tau'_{\alpha,0}\tau_{\beta,0}, \tag{148}$$

$$W^{(1)RS}(\alpha, \beta|v', v)_{1,L;1,L} = f_+^R(v + v' + 2K + 2L\eta)g_-^R(v - v' + 2L\eta)\tau'_{\alpha,0}\tau_{\beta,L}, \tag{149}$$

$$W^{(1)RS}(\alpha, \beta|v', v)_{L,1;L,1} = f_-(v + v' - 2L\eta)g_-^R(v - v' + 2L\eta)\tau'_{\alpha,L}\tau_{\beta,0}, \tag{150}$$

$$W^{(1)RS}(\alpha, \beta|v', v)_{L,L;L,L} = f_-^R(v + v' + 2K)g_-^R(v - v' + 4L\eta)\tau'_{\alpha,L}\tau_{\beta,L}. \tag{151}$$

When $N$ is odd, the antisymmetry of $g_-^R(z)$ in (148) prevents the proof of [1] from being used. However, when $N$ is even the operators $S$ and $R$ commute and thus (49) with $A = RS$

**Table 1.** Summary of the values of the matrix $A$ for which the interchange relation (49) holds.

| $m_1$ | $m_2$ | $I$ | $S$ | $R$ | $RS$ |
|-------|-------|-----|-----|-----|------|
| o | e | Y | Y | N | N |
| o | o | N | Y | Y | N |
| e | o | Y | N | Y | N |
| e | e | Y | Y | Y | Y |

holds if the equivalent relation

$$Q_L(v_1)RSQ_R(v_2) = Q_L(v_2)SRQ_R(v_1) \tag{152}$$

is valid. But

$$Q_L^{(1)}(v')SRQ_R^{(1)}(v)|_{\alpha,\beta} = \text{Tr } W^{(1)SR}(\alpha_1, \beta_1|v', v) \cdots W^{(1)SR}(\alpha_N, \beta_N|v', v) \tag{153}$$

where

$$W^{(1)SR}(\alpha, \beta|v', v)_{k,k';l,l'} = \sum_{\gamma=\pm}(-\gamma)S_L^{(1)}(\alpha, \gamma|v')_{k,l}RS_R^{(1)}(-\gamma, \beta|v)_{k',l'}$$

$$= -W^{(1)SR}(\alpha, \beta|v', v)_{k,k';l,l'} \tag{154}$$

and the minus sign in (154) compensates for the antisymmetry of $g_-^R(z)$. Therefore, by the use of (146) in the case

$$(-1)^{m_1+m_2}(-1)^{m_1m_2} = 1, \tag{155}$$

we find that (152) holds and hence we find that the interchange relation (49) holds for $N$ even in the case $m_1$ and $m_2$ even but fails in the other three cases.

*5.1.5. Summary.* The results obtained above for the validity of the interchange relation (49) for $A = I, S, R$ and $RS$ are summarized in table 1, where Y (N) indicates that the relation holds (fails).

### 5.2. The nonsingularity condition.

It remains to examine the validity of the nonsingularity condition. In the case of $m_2 = 0$, this condition was numerically studied in [14] for several values of $L$ and $N$ and it was found that for $L$ odd that $Q_R(v)$ was nonsingular for all $v$ only when $m_1$ was odd. We have extended that study to $m_2 \neq 0$ and found that for the cases studied $Q_R(v)$ is singular for all $v$ only when $L$ is odd and both $m_1$ and $m_2$ are even.

In the remaining three cases where one or both of $m_1$ and $m_2$ are odd the matrices $Q_R(v)$ and $Q_L(v)$ were nonsingular for generic values of $v$; we conjecture that this is generally true.

### 5.3. The matrices $Q_{72}^{(1)}(v)$

Using the results for the interchange relation summarized in table 1 and assuming the validity of the conjecture of section 5.2 on the nonsingularity of $Q_R(v)$ we conclude that the matrix $Q_{72}^{(1)}(v)$ defined by (51) satisfies the $TQ$ equation (1) and the commutation relations (2) and (3) for both case 1 where $m_1$ is odd and $m_2$ is even and case 3 where $m_1$ is even and $m_2$ is odd. It further follows from the relations summarized in table 1 that the commutation relations (9)–(11) with $R, S$ and $RS$ hold for the three cases where $m_1$ and $m_2$ are not both even.

## 6. The matrix $Q^{(2)}_{72ee}(v; t)$ for case 4 where $m_1, m_2$ and $N$ is even

We found in section 5.2 that in case 4 where $L$ is odd and $m_1, m_2$ are even the matrix $Q^{(1)}_{72ee}$ does not exist. Therefore, to satisfy the $TQ$ equation

$$T(v)Q^{(2)}(v; t) = \omega^N[h(v + \eta)]^N Q^{(2)}(v - 2\eta; t) + \omega^{-N}[h(v - \eta)]^N Q^{(2)}(v + 2\eta; t) \qquad (156)$$

and the commutation relations (2)–(3), a new construction must be found. For $m_2 = 0$ this was accomplished in [16]. We here generalize this construction to even values of $m_2 \neq 0$. In section 4.2, we demonstrated that the matrix $Q^{(2)}_R(v)$ defined by the matrices $S^{(2)}_R(v; t)$ of (78) satisfies equation (79) and that there is a companion equation for $Q^{(2)}_L(v)$. Therefore, to complete the proof of the $TQ$ equation (156), we must find the values of $t$ and matrices $A$ for which

$$Q^{(2)}_L(v'; t)AQ^{(2)}_R(v; t) = Q^{(2)}_L(v; t)AQ^{(2)}_R(v'; t) \qquad (157)$$

holds for which $Q^{(2)}_R(v; t)$ is nonsingular.

### 6.1. The interchange relations

We consider the cases of $A = I$, $S$, $R$ and $RS$ separately.

*6.1.1. The case $A = I$.* We begin by examining (157) with $A = I$ and write

$$Q^{(2)}_L(v'; t)Q^{(2)}_R(v; t)|_{\alpha,\beta} = \text{Tr } W^{(2)}(\alpha_1, \beta_1|v', v) \cdots W^{(2)}(\alpha_N, \beta_N|v', v), \qquad (158)$$

where $W^{(2)}(\alpha, \beta|v', v)$ are $L^2 \times L^2$ matrices with elements

$$W^{(2)}(\alpha, \beta|v', v)_{k,k';l,l'} = \sum_{\gamma=\pm} S^{(2)}_L(\alpha, \gamma|v')_{k,l} S^{(2)}_R(\gamma, \beta|v)_{k',l'}. \qquad (159)$$

The matrix $W^{(2)}(\alpha, \beta|v', v)$ is explicitly written out as

$$W^{(2)}_{k,k';k+1,k'+1}(\alpha, \beta|v', v) = \tau'_{\alpha,-k}\tau_{\beta,-k'}f_-(v' + v + 2(k - k')\eta)g_-(v' - v + 2t + 2(k + k')\eta), \qquad (160)$$

$$W^{(2)}_{k+1,k'+1;k,k'}(\alpha, \beta|v', v) = \tau'_{\alpha,k}\tau_{\beta,k'}f_-(v' + v - 2(k - k')\eta)g_-(v' - v - 2t - 2(k + k')\eta), \qquad (161)$$

$$W^{(2)}_{k,k'+1;k+1,k'}(\alpha, \beta|v', v) = \tau'_{\alpha,-k}\tau_{\beta,k'}f_+(v' + v + 2t + 2(k + k')\eta)g_+(v' - v + 2(k - k')\eta), \qquad (162)$$

$$W^{(2)}_{k+1,k';k,k'+1}(\alpha, \beta|v', v) = \tau'_{\alpha,k}\tau_{\beta,-k'}f_+(v' + v - 2t - 2(k + k')\eta)g_+(v' - v - 2(k - k')\eta), \qquad (163)$$

where $f_+(z)$ and $g_+(z)$ are given by (89) and (90) and $f_-(z)$ and $g_-(z)$ are given by (109) and (110), respectively.

We again look for an $L^2 \times L^2$ diagonal matrix $Y^{(2)}$

$$Y_{m,m';k,k'} = y_{m,m'}\delta_{m,k}\delta_{m',k'} \qquad (164)$$

such that

$$W^{(2)}(\alpha, \beta|v', v) = Y^{(2)}W^{(2)}(\alpha, \beta|v, v')Y^{(2)-1}. \qquad (165)$$

The expressions for the diagonal elements are symmetric under the interchange of $v'$ and $v$ and thus there are no restrictions such as we had for $Q_{72}^{(1)}$. Using (160), (161) and $g_-(z) = g_-(-z)$ in (164) and (165) we find the single equation

$$g_-(v' - v + 2t + 2(k + k')\eta) = \frac{y_{k,k'}}{y_{k+1,k'+1}} g_-(v - v' + 2t + 2(k + k')\eta), \quad (166)$$

or equivalently

$$y_{k+1,k'+1} = y_{k,k'} \frac{g_-(v' - v - 2(k + k')\eta - 2t)}{g_-(v' - v + 2(k + k')\eta + 2t)}. \quad (167)$$

Similarly, by using (162), (163) and $g_+(z) = g_+(-z)$ in (164) and (165) we find the single equation

$$g_+(v' - v + 2(k - k')\eta) = \frac{y_{k,k'+1}}{y_{k+1,k'}} g_+(v - v' + 2(k - k')\eta), \quad (168)$$

and thus we obtain

$$y_{k+1,k'-1} = y_{k,k'} \frac{g_+(v - v' + 2(k - k' + 1)\eta)}{g_+(v - v' - 2(k - k' + 1)\eta)}. \quad (169)$$

We follow [16] and note that for the recursions (167) to be free of contradictions we need

$$y_{k+L,k'+L} = y_{k,k'} \quad (170)$$

and for (169) to be free of contradictions

$$y_{k-L,k'+L} = y_{k,k'}. \quad (171)$$

In order for (170) to hold, we need to choose $t$ such that

$$\frac{g_-(v - v' - 2(k + k')\eta - 4(L - 1)\eta - 2t)}{g_-(v - v' + 2(k + k')\eta + 4(L - 1)\eta + 2t)} \frac{g_-(v - v' - 2(k + k')\eta - 4(L - 2)\eta - 2t)}{g_-(v - v' + 2(k + k')\eta + 4(L - 2)\eta + 2t)} \cdots$$
$$\frac{g_-(v - v' - 2(k + k')\eta - 2t)}{g_-(v - v' + 2(k + k')\eta + 2t)} = 1 \quad (172)$$

which will be satisfied if the factors $g_-(v - v' - 2(k + k')\eta - 4c_1\eta - 2t)$ in the numerator must cancel the factors $g_-(v - v' + 2(k + k')\eta + 4c_2\eta + 2t)$ in the denominator. For this we need to use the periodicity properties of $g_-(v)$.

From the definition (110) of $g_-(v)$ and the periodicity of $H_m(v)$ (28), (32), it follows that for all even $m_1$ and $m_2$

$$g_-(v + 4(r_1 K + ir_2 K')) = g_-(v). \quad (173)$$

Furthermore, we use the definitions (4) and (34) in (114) to find

$$g_-(z + r_0(r_1 K + ir_2 K')) = (-1)^{m_1 m_2/4}(-1)^{m_2/2} g_-(z). \quad (174)$$

When $m_2 \equiv 2 \pmod 4$ we see from (34) that $r_0 \equiv 2 \pmod 4$ and thus it follows from (173) and (174) that we have the additional periodicity condition

$$g_-(z + 2(r_1 K + ir_2 K')) = -(-1)^{m_1/2} g_-(z). \quad (175)$$

Consider first the periodicity (173). This will provide cancellation if an integer $I$ can be found such that

$$-2(k + k')\eta - 4c_1\eta - 2t = 2(k + k')\eta + 4c_2\eta + 2t + 4I(r_1 K + ir_2 K') \quad (176)$$

which by multiplying by $L$, using (4) and (34) and defining

$$t = \bar{t}\eta, \quad (177)$$

becomes

$$-(k + k')r_0 - c_1 r_0 - \bar{t} r_0 = c_2 r_0 + 2LI. \tag{178}$$

For $m_2$ even the quantity $r_0$ is always even and thus (178) can always be satisfied by integers for $\bar{t} = n$ with $n$ an integer because $c_2$ can be shifted into the interval $0 \leqslant c_2 < L$. Furthermore, if $m_2 \equiv 0 (\mathrm{mod}\, 4)$ then for even $m_1$ we have $r_0 \equiv 0 (\mathrm{mod}\, 4)$ and thus (178) may be satisfied by integers for $\bar{t} = n + 1/2$. Thus, we have demonstrated that for all cases of $m_1$ and $m_2$ even that (170) holds for $\bar{t} = n$ and for $m_2 \equiv 0 (\mathrm{mod}\, 4)$ and $m_1$ even that (157) with $A = 1$ is satisfied for $\bar{t} = n + 1/2$ but is not satisfied if $m_2 \equiv 2 (\mathrm{mod}\, 4)$.

We next consider the periodicity condition (175) which holds for $m_2 \equiv 2 (\mathrm{mod}\, 4)$ and $r_0 \equiv 2 (\mathrm{mod}\, 4)$ which with the additional restriction that $m_1 \equiv 2 (\mathrm{mod}\, 4)$ specializes to

$$g_-(z + 2(r_1 K + i r_2 K')) = g_-(z). \tag{179}$$

This will give the desired cancellation in (172) if instead of (178) we have

$$-(k + k')r_0 - c_1 r_0 - \bar{t} r_0 = c_2 r_0 + LI. \tag{180}$$

Using the fact that $r_0 \equiv 2 (\mathrm{mod}\, 4)$, we see that this equation can be satisfied in integers for $\bar{t} = n + 1/2$. Thus, we have demonstrated that (170) is satisfied for the case $m_1 \equiv 2 (\mathrm{mod}\, 4)$ and $m_2 \equiv 2 (\mathrm{mod}\, 4)$ with $t = (n + 1/2)\eta$.

To complete the proof of (157) with $A = 1$ it remains to demonstrate that (171) holds. We see from (169) that this will be the case

$$\frac{g_+(v - v' + 2(k - k' + 1)\eta - 4(L - 1)\eta)}{g_+(v - v' - 2(k - k' + 1)\eta + 4(L - 1)\eta)} \frac{g_+(v - v' + 2(k - k' + 1)\eta - 4(L - 2)\eta)}{g_+(v - v' - 2(k - k' + 1)\eta + 4(L - 2)\eta)}$$
$$\frac{g_+(v - v' + 2(k - k' + 1)\eta)}{g_+(v - v' - 2(k - k' + 1)\eta)} = 1. \tag{181}$$

In contrast to (172) this is independent of $t$ and therefore, using the periodicity condition which follows from the definition (90) of $g_+(z)$ and the periodicity of $H_m(v)$ (28), (32)

$$g_+(v + 4(r_1 K + i r_2 K')) = g_+(v), \tag{182}$$

we show that (181) holds for all $m_1$ and $m_2$ even. Thus, we have proven that (157) with $A = I$ holds for the cases

$$m_1 \text{ even}, \qquad m_2 \text{ even with} \quad t = n\eta, \tag{183}$$

$$m_1 \text{ even}, \qquad m_2 \equiv 0 (\mathrm{mod}\, 4) \text{ with} \quad t = (n + 1/2)\eta, \tag{184}$$

$$m_1 \equiv 2 (\mathrm{mod}\, 4), \qquad m_2 \equiv 2 (\mathrm{mod}\, 4) \text{ with} \quad t = (n + 1/2)\eta. \tag{185}$$

*6.1.2. The case $A = S$.* We study the interchange relation (157) with $A = S$ in a completely analogous manner by writing

$$Q_L^{(2)}(v'; t) S Q_R^{(2)}(v; t)|_{\alpha, \beta} = \mathrm{Tr}\, W^{(2S)}(\alpha_1, \beta_1 | v', v) \cdots W^{(2S)}(\alpha_N, \beta_N | v', v), \tag{186}$$

where $W^{(2S)}(\alpha, \beta | v', v)$ are $L^2 \times L^2$ matrices with elements

$$W^{(2S)}(\alpha, \beta | v', v)_{k, k'; l, l'} = \sum_{\gamma = \pm} S_L^{(2)}(\alpha, \gamma | v')_{k, l} \gamma S_R^{(2)}(\gamma, \beta | v)_{k', l'} \tag{187}$$

is explicitly written out as

$$W_{k, k'; k+1, k'+1}^{(2S)}(\alpha, \beta | v', v) = \tau'_{\alpha, -k} \tau_{\beta, -k'} f_+(v' + v + 2(k - k')\eta) g_+(v' - v + 2t + 2(k + k')\eta), \tag{188}$$

$$W^{(2S)}_{k+1,k'+1;k,k'}(\alpha, \beta|v', v) = \tau'_{\alpha,k}\tau_{\beta,k'}f_+(v' + v - 2(k - k')\eta)g_+(v' - v - 2t - 2(k + k')\eta), \tag{189}$$

$$W^{(2S)}_{k,k'+1;k+1,k'}(\alpha, \beta|v', v) = \tau'_{\alpha,-k}\tau_{\beta,k'}f_-(v' + v + 2t + 2(k + k')\eta)g_-(v' - v + 2(k - k')\eta), \tag{190}$$

$$W^{(2S)}_{k+1,k';k,k'+1}(\alpha, \beta|v', v) = \tau'_{\alpha,k}\tau_{\beta,-k'}f_-(v' + v - 2t - 2(k + k')\eta)g_-(v' - v - 2(k - k')\eta), \tag{191}$$

where the roles of $f_+(z)$ and $g_+(z)$ and $f_-(z)$ and $g_-(z)$ are reversed from (160)–(163).

We now use the periodicity property

$$g_+(z + r_0(r_1K + \mathrm{i}r_2K')) = (-1)^{m_1m_2/4}g_+(z), \tag{192}$$

which follows from (94) and (182) in exactly the same manner used to prove (157) with $A = I$ to prove that (157) with $A = S$

$$m_1 \text{ even}, \qquad m_2\text{even with} \quad t = n\eta, \tag{193}$$

$$m_1 \text{ even}, \qquad m_2 \equiv 0(\mathrm{mod}\,4) \text{ with} \quad t = (n + 1/2)\eta, \tag{194}$$

$$m_1 \equiv 0(\mathrm{mod}\,4), \qquad m_2 \equiv 2(\mathrm{mod}\,4) \text{ with} \quad t = (n + 1/2)\eta. \tag{195}$$

*6.1.3. The case $A = R$.*   To investigate (157) with $A = R$, we write

$$Q^{(2)}_L(v'; t)RQ^{(2)}_R(v; t)|_{\alpha,\beta} = \mathrm{Tr}\,W^{(2R)}(\alpha_1, \beta_1|v', v)\cdots W^{(2R)}(\alpha_N, \beta_N|v', v), \tag{196}$$

where $W^{(2R)}(\alpha, \beta|v', v)$ are $L^2 \times L^2$ matrices with elements

$$W^{(2R)}(\alpha, \beta|v', v)_{k,k';l,l'} = \sum_{\gamma=\pm} S^{(2)}_L(\alpha, \gamma|v')_{k,l}S^{(2)}_R(-\gamma, \beta|v)_{k',l'}. \tag{197}$$

By use of the identities (125) and (142), the matrix $W^{(2)R}(\alpha, \beta|v', v)$ is explicitly written out as

$$W^{(2R)}_{k,k';k+1,k'+1}(\alpha, \beta|v', v) = \tau'_{\alpha,-k}\tau_{\beta,-k'}f^R_-(v + v' + 2(k - k')\eta)g^R_-(v' - v + 2t + 2(k + k')\eta), \tag{198}$$

$$W^{(2R)}_{k+1,k'+1;k,k'}(\alpha, \beta|v', v) = \tau'_{\alpha,k}\tau_{\beta,k'}f^R_-(v + v' - 2(k - k')\eta)g^R_-(v' - v - 2t - 2(k + k')\eta), \tag{199}$$

$$W^{(2R)}_{k,k'+1;k+1,k'}(\alpha, \beta|v', v) = \tau'_{\alpha,-k}\tau_{\beta,k'}f^R_+(v + v' + 2t + 2(k + k')\eta)g^R_+(v' - v - 2(k - k')\eta), \tag{200}$$

$$W^{(2R)}_{k+1,k';k,k'+1}(\alpha, \beta|v', v) = -\tau'_{\alpha,k}\tau_{\beta,-k'}f^R_+(v + v' - 2t - 2(k + k')\eta)g^R_+(v' - v - 2(k - k')\eta). \tag{201}$$

We again follow the procedure of the previous subsection and look for a diagonal similarity transformation. However, because of the antisymmetry of $g^R_-(z)$ and recalling that the $L$ is odd the consistency condition analogous to (172) for (198) and (199) is

$$\frac{g^R_-(v - v' - 2(k + k')\eta - 4(L - 1)\eta - 2t)}{g^R_-(v - v' + 2(k + k')\eta + 4(L - 1)\eta + 2t)}\frac{g^R_-(v - v' - 2(k + k')\eta - 4(L - 2)\eta - 2t)}{g^R_-(v - v' + 2(k + k')\eta + 4(L - 2)\eta + 2t)}\cdots$$
$$\frac{g^R_-(v - v' - 2(k + k')\eta - 2t)}{g^R_-(v - v' + 2(k + k')\eta + 2t)} = -1, \tag{202}$$

whereas in analogy with (181) the consistency condition for (200) and (201) is

$$\frac{g_+^R(v - v' + 2(k - k' + 1)\eta - 4(L - 1)\eta)}{g_+^R(v - v' - 2(k - k' + 1)\eta + 4(L - 1)\eta)} \frac{g_+^R(v - v' + 2(k - k' + 1)\eta - 4(L - 2)\eta)}{g_+^R(v - v' - 2(k - k' + 1)\eta + 4(L - 2)\eta)} \cdots$$

$$\frac{g_+^R(v - v' + 2(k - k' + 1)\eta)}{g_+^R(v - v' - 2(k - k' + 1)\eta)} = 1. \tag{203}$$

We see that we will satisfy (202) if there is the pairwise relation between factors in the numerator and denominator of

$$g_-^R(v - v' - 2(k + k')\eta - 4c_1\eta - 2t) = -g_-^R(v - v' + 2(k + k')\eta + 4c_2\eta + 2t) \tag{204}$$

and therefore in contrast with the cases $A = I$ and $S$ we must consider antiperiodicity properties of $g_-^R(z)$

In contrast, the condition (203) is satisfied if

$$g_+^R(v - v' + 2(k - k' + 1)\eta - 4c_1\eta) = g_+^R(v - v' - 2(k - k' + 1)\eta + 4c_1\eta) \tag{205}$$

which requires a periodicity (and not an antiperiodicity) property for $g_+^R(z)$.

The (anti)periodicity properties analogous to (173) follow from the definitions (127) of $g_+^R(z)$ and (144) of $g_-^R(z)$ and the properties (28) and (29) of $H_m(z)$ and $\Theta_m(z)$

$$g_\pm^R(z + 4(r_1 K + ir_2 K')) = (-1)^{r_1} g_\pm^R(z). \tag{206}$$

Consider first the condition (205). In the case that $r_1$ is even we have from (206)

$$g_+^R(z + 4(r_1 K + ir_2 K')) = g_+^R(z), \tag{207}$$

and thus (205) will hold if

$$4(k - k' + 1)\eta - 4c_1 = 4c_2 + 4I(r_1 K + ir_2 K'), \tag{208}$$

which after we multiply by $L$ and use the root of unity condition (4) becomes

$$(k - k' + 1)r_0 - c_1 r_0 = c_2 r_0 + 2LI \tag{209}$$

which can always be satisfied in integers because for $m_1$ and $m_2$ even $r_0$ is always even.

In the opposite case, where $r_1$ is odd we have from (206)

$$g_+^R(z + 8(r_1 K + ir_2 K')) = g_+^R(z) \tag{210}$$

and thus (205) will be satisfied if

$$(k - k' + 1)r_0 - c_1 r_0 = c_2 r_0 + 4LI. \tag{211}$$

This can only hold if $r_0 \equiv 0 \pmod 4$. However, it follows from the definition (34) of $r_1$ that when $m_1$ is even and $r_1$ is odd that $r_0/4$ must be an integer. Therefore, the condition (205) is always satisfied.

We next attempt to satisfy (204) by use of (206) for $r_1$ odd. Thus,

$$g_-^R(z + 4(2I + 1)(r_1 K + ir_2 K')) = -g_-^R(z) \tag{212}$$

and (204) will be satisfied if

$$-(k + k')r_0 - c_1 r_0 - r_0 \bar{t} = c_2 r_0 + 2(2I + 1)L. \tag{213}$$

However, we have just seen that when $r_1$ is odd and $m_1$ is even that $r_0/4$ and $m_2/4$ must be integers. Therefore, because $2I + 1$ and $L$ are odd (213) cannot be satisfied in integers for any integer $\bar{t}$. Therefore, there are no solutions for $A = R$ which are analogous to the solutions $t = n\eta$ for the cases $A = I$ and $S$. However, if $\bar{t} = n + 1/2$ the condition (213) can be satisfied

**Table 2.** Summary of the values of the matrix $A$ for which the interchange relation (157) with $t = n\eta$ holds where the notation 0(2) stands for $\equiv 0(2)(\mathrm{mod}\,4)$.

| $m_1$ | $m_2$ | $I$ | $S$ | $R$ | $RS$ |
|-------|-------|-----|-----|-----|------|
| 0 | 0 | Y | Y | N | N |
| 2 | 0 | Y | Y | N | N |
| 0 | 2 | Y | Y | N | N |
| 2 | 2 | Y | Y | N | N |

**Table 3.** Summary of the values of the matrix $A$ for which the interchange relation (157) with $t = (n + 1/2)\eta$ holds where the notation 0(2) stands for $\equiv 0(2)(\mathrm{mod}\,4)$.

| $m_1$ | $m_2$ | $I$ | $S$ | $R$ | $RS$ |
|-------|-------|-----|-----|-----|------|
| 0 | 0 | Y | Y | N | N |
| 2 | 0 | Y | Y | Y | Y |
| 0 | 2 | N | Y | Y | N |
| 2 | 2 | Y | N | Y | N |

in integers if $r_0/4$ is an odd integer which leads to the conclusion that the interchange relation (157) with $A = R$ holds for $t = (n + 1/2)\eta$ when

$$m_1 \equiv 2(\mathrm{mod}\,4), \qquad m_2 \equiv 0(\mathrm{mod}\,4). \tag{214}$$

It remains to investigate the possibility of using (147) to satisfy (204). The condition (147) is antiperiodic in the three cases where $m_1/4$ and $m_2/4$ are not both integers. We have already considered the case $m_2 \equiv 0(\mathrm{mod}\,4)$ and thus need only consider the remaining cases where $m_2 \equiv 2(\mathrm{mod}\,4)$. In these cases $r_0 \equiv 2(\mathrm{mod}\,4)$ and thus, because $m_1$ is even $r_1$ must be even. Thus, we find from (206) and (147) that for $m_1$ even and $m_2 \equiv 2(\mathrm{mod}\,4)$ that

$$g_-^R(z + 2(2I + 1)(r_1 K + \mathrm{i}r_2 K')) = -g_-^R(z) \tag{215}$$

and from this it follows that (202) holds for $t = (n + 1/2)\eta$.

*6.1.4. The case $A = RS$.* The case $A = RS$ is treated by similar methods and we find that the only case where the interchange relation (157) holds is for $t = (n + 1/2)\eta$, $m_1 \equiv 2(\mathrm{mod}4)$ and $m_2 \equiv 0(\mathrm{mod}\,4)$.

*6.1.5. Summary.* The results obtained above for the validity of the interchange relation (157) with $t = n\eta$ and $(n + 1/2)\eta$ for $A = I$, $S$, $R$ and $RS$ are summarized in tables 2 and 3, respectively, where Y (N) indicates that the relation holds (fails).

*6.2. The nonsingularity condition*

We have found numerically for several special cases that $Q_R^{(2)}(v; n\eta)$ is nonsingular for $m_1$ and $m_2$ even and that $Q_R^{(2)}(v; (n + 1/2)\eta)$ is singular only for $m_1 \equiv 2(\mathrm{mod}\,4)$ and $m_2 \equiv 0(\mathrm{mod}\,4)$. We conjecture that this is true generally.

*6.3. The matrices $Q_{72ee}^{(2)}(v; t)$ for $t = n\eta$ and $(n + 1/2)\eta$*

From the results for the interchange relation (157) summarized in table 2 for $t = n\eta$ and in table 3 for $t = (n + 1/2)\eta$ and assuming the validity of the nonsingularity conjecture we

conclude that the matrix $Q_{72}^{(2)}(v; n\eta)$ is constructed from (51) for all even $m_1$ and $m_2$ with the commutation relation given in (12) of the introduction and that the matrix $Q_{72}^{(2)}(v; (n + 1/2)\eta)$ is similarly constructed from (51) for the three cases

$$
\begin{aligned}
m_1 &\equiv 0 \pmod 4, & m_2 &\equiv 0 \pmod 4 \\
m_1 &\equiv 2 \pmod 4, & m_2 &\equiv 2 \pmod 4 \\
m_1 &\equiv 0 \pmod 4, & m_2 &\equiv 2 \pmod 4
\end{aligned}
\tag{216}
$$

with the commutation relations given by (13)–(15).

## 7. Quasiperiodicity properties of $Q_{72}^{(1)}(v)$ and $Q_{72}^{(2)}(v; t)$

We complete our discussion of the $Q$ matrices by deriving their quasiperiodicity properties and general form of the eigenvalues.

### 7.1. Quasiperiodicity of $Q_{72}^{(1)}(v)$

To compute the quasiperiodicity properties of $Q_{72}^{(1)}(v)$, we first use the quasiperiodicity properties (28)–(31) in the definition of $S_R^{(1)}$ (68) to find

$$
S_R^{(1)}(\alpha, \beta)_{j,k}(v + \omega_1) = (-\alpha)^{r_1}(-1)^{r_1 r_2} S_R^{(1)}(\alpha, \beta)_{j,k}(v)
\tag{217}
$$

and

$$
\begin{aligned}
S_R^{(1)}(\alpha, \beta)_{k,k+1}(v + \omega_2) &= (-\alpha)^b (-1)^{ab} q'^{-1} \mathrm{e}^{-2\pi \mathrm{i} v/\omega_1} \\
&\quad \times \mathrm{e}^{4\pi \mathrm{i} k\eta/\omega_1} S_R^{(1)}(\alpha, \beta)_{k,k+1}(v) \qquad 1 \leqslant k \leqslant L - 1 \\
S_R^{(1)}(\alpha, \beta)_{k+1,k}(v + \omega_2) &= (-\alpha)^b (-1)^{ab} q'^{-1} \mathrm{e}^{-2\pi \mathrm{i} v/\omega_1} \\
&\quad \times \mathrm{e}^{-4\pi \mathrm{i} k\eta/\omega_1} S_R^{(1)}(\alpha, \beta)_{k+1,k}(v) \qquad 1 \leqslant k \leqslant L - 1 \\
S_R^{(1)}(\alpha, \beta)_{1,1}(v + \omega_2) &= (-\alpha)^b (-1)^{ab} q'^{-1} \mathrm{e}^{-2\pi \mathrm{i} v/\omega_1} S_R^{(1)}(\alpha, \beta)_{1,1}(v) \\
S_R^{(1)}(\alpha, \beta)_{L,L}(v + \omega_2) &= (-\alpha)^b (-1)^{ab} (-1)^{r_0} q'^{-1} \mathrm{e}^{-2\pi \mathrm{i} v/\omega_1} S_R^{(1)}(\alpha, \beta)_{L,L}(v).
\end{aligned}
\tag{218}
$$

The dependence of (218) on $r_0$ distinguishes case 1 with $m_1$ odd and $m_2$ even from cases 2 and 3 with $m_2$ odd.

#### 7.1.1. Case 1 with $m_1$ odd and $m_2$ even.
In case 1 where $m_1$ is odd and $m_2$ is even, the greatest common factor $r_0$ in $2m_1$ and $m_2$ is even, $r_1$ must be odd but $r_2$ is unrestricted. Therefore, we find directly from the quasiperiodicity (217) and from (38) and (51) that

$$
Q_{72oe}^{(1)}(v + \omega_1) = -S(-1)^{Nr_2} Q_{72oe}^{(1)}(v).
\tag{219}
$$

To examine quasiperiodicity under $v \rightarrow v + \omega_2$, we use the diagonal similarity transformation (2.13) of [14] to write (218) as

$$
S_R^{(1)}(\alpha, \beta)(v + \omega_2) = (-\alpha)^b (-1)^{ab} q'^{-1} \mathrm{e}^{-2\pi \mathrm{i} v/\omega_1} M^{(1)} S_R^{(1)}(\alpha, \beta)(v) M^{(1)-1},
\tag{220}
$$

with

$$
M_{k,k'}^{(1)} = \mathrm{e}^{-2\pi \mathrm{i} \eta k(k-1)/\omega_1} \delta_{k,k'}.
\tag{221}
$$

Thus, we find from (38) and (51) that

$$
Q_{72oe}^{(1)}(v + \omega_2) = (-S)^b (-1)^{Nab} \mathrm{e}^{-2\pi \mathrm{i} Nv/\omega_1} q'^{-N} Q_{72oe}^{(1)}(v).
\tag{222}
$$

It follows from (219), (222) and the fact that the eigenvectors of $Q_{72}^{(1)}(v)$ are independent of $v$ that $Q_{72oe}^{(1)}(v)$ commutes with $S$ (as was shown directly in section 5).

It follows from (219) and (222) that $Q_{72oe}^{(1)}(v)$ has $N$ zeros in the fundamental parallelogram (35)

$$0, \omega_1, \omega_1 + \omega_2, \omega_2 \tag{223}$$

and thus may be written in factorized form as

$$Q_{72oe}^{(1)}(v) = \mathcal{K} \exp(-\mathrm{i}\nu\pi v/\omega_1) \prod_{j=1}^{N} \mathrm{H}_m(v - v_j) \tag{224}$$

with $v_j$ in the parallelogram (223). Using the form (224) we find from the quasiperiodicity condition (219)

$$1 = \mathrm{e}^{\pi \, \mathrm{i}(1+\nu_S+\nu+N)}, \tag{225}$$

and thus

$$1 + \nu + N + \nu_S = \text{even integer} \tag{226}$$

where $(-1)^{\nu_S}$ is the eigenvalue of $S$.

From (222) we find

$$(-S)^b = \mathrm{e}^{-\nu\pi \, \mathrm{i}\omega_2/\omega_1}(-1)^{bN} \exp\left(2\pi\mathrm{i}\sum_{j=1}^{N}(v_j + K)/\omega_1\right), \tag{227}$$

and thus

$$b(\nu_S + 1) - \nu\omega_2/\omega_1 + bN + 2\sum_{j=1}^{N}(v_j + K)/\omega_1 = \text{even integer}. \tag{228}$$

*7.1.2. Cases 2 and 3 with $m_2$ odd.* When $m_2$ is odd we see from (34) that $r_0$ and $r_2$ are odd and $r_1$ is even for both $m_1$ even and odd. Therefore, because $r_0$ is odd we find instead of the quasiperiodicity condition (220) under $v \to v + \omega_2$ we have instead a quasiperiodicity under $v \to v + 2\omega_2$

$$S_R^{(1)}(\alpha, \beta)(v + 2\omega_2) = q'^{-4} \, \mathrm{e}^{-4\pi \, \mathrm{i}v/\omega_1} M^{(1)2} S_R^{(1)}(\alpha, \beta)(v) M^{(1)-2} \tag{229}$$

to find

$$Q_R^{(1)}(v + 2\omega_2) = q'^{-4N} \, \mathrm{e}^{-4\pi \, \mathrm{i}Nv/\omega_1} Q_R^{(1)}(v). \tag{230}$$

Thus, by use of (51) we find

$$Q_{72xo}^{(1)}(v + 2\omega_2) = q'^{-4N} \, \mathrm{e}^{-4\pi \, \mathrm{i}Nv/\omega_1} Q_{72xo}^{(1)}(v) \tag{231}$$

where $x$ is either e or o.

If the area of the fundamental parallelogram is to be $4KK'$, then the quasiperiodic property (231) mandates that instead of the parallelogram (223) we need to consider the parallelogram

$$0, \omega_1/2, \omega_1/2 + 2\omega_2, 2\omega_2. \tag{232}$$

To obtain the periodicity properties under $v \to \omega_1/2$ we write

$$\omega_1/2 = r_1 K + \mathrm{i}r_2 K', \tag{233}$$

where $r_1/2$ is an integer because $r_1$ is even. It then follows from the definitions (25) and the properties (A.5), (A.8) and (A.9) that

$$\mathrm{H}_m(v + \omega_1/2) = (-1)^{r_1/2}\, \mathrm{e}^{\pi\, \mathrm{i} r_1 r_2/4}\, \Theta_m(v), \tag{234}$$

$$\Theta_m(v + \omega_1/2) = \mathrm{e}^{\pi\, \mathrm{i} r_1 r_2/4}\, \mathrm{H}_m(v). \tag{235}$$

We therefore obtain for $m_2$ odd and all $m_1$ that

$$S_R^{(1)}(v + \omega_1/2) = \mathrm{e}^{\pi\, \mathrm{i} r_1 r_2/4}\, R S^{r_1/2}\, S_R^{(1)}(v). \tag{236}$$

*7.1.3. Case 3 with $m_1$ even.*   When $m_1$ is further restricted to be even we find from (34) that $r_1/2$ is even and therefore (236) may be written as

$$S_R^{(1)}(v + \omega_1/2) = \mathrm{e}^{\pi\, \mathrm{i} r_1 r_2/4}\, R S_R^{(1)}(v). \tag{237}$$

Therefore, we find from (38) and (51) that

$$Q_{72eo}^{(1)}(v + \omega_1/2) = \mathrm{e}^{N\pi\, \mathrm{i} r_1 r_2/4}\, R Q_{72eo}^{(1)}(v) \tag{238}$$

and from (238) and the fact that the eigenvectors of $Q_{72eo}^{(1)}(v)$ are independent of $v$ it follows that

$$\left[ Q_{72eo}^{(1)}(v), R \right] = 0 \tag{239}$$

which has been directly proven in section 5.

From the quasiperiodicity relations (231) and (238), it follows that the eigenvalues of $Q_{72eo}^{(1)}(v)$ may be written in terms of $\mathrm{H}_m(v)$ as

$$Q_{72eo}^{(1)}(v) = \mathcal{K}\, \mathrm{e}^{-v 2\pi\, \mathrm{i} v/\omega_1} \prod_{j=1}^{N} \mathrm{H}_m(v/2 - v_j/2)\, \mathrm{H}_m(v/2 - v_j/2 + \omega_1/4)$$
$$\times\, \mathrm{H}_m(v/2 - v_j/2 + \omega_1/2)\, \mathrm{H}_m(v/2 - v_j/2 + 3\omega_1/4), \tag{240}$$

where the $N$ roots $v_j$ lie in the fundamental parallelogram (232). From the quasiperiodicity relation (238), we find the sum rule

$$\mathrm{e}^{N\pi\, \mathrm{i} r_1 r_2/4} (-1)^{v_R} = (-1)^v \tag{241}$$

where $(-1)^{v_R}$ are the eigenvalues of $R$ and from the quasiperiodicity relation (231), we find the sum rule

$$1 = \exp\left( -v 4\pi\, \mathrm{i}\omega_2/\omega_1 + 4\pi\, \mathrm{i} \sum_{j=1}^{N} (v_j + 2K)/\omega_1 \right). \tag{242}$$

*7.1.4. Case 2 with $m_1$ odd.*   The quasiperiodicity relation (236) also holds but now for $m_1$ odd we have from (34) that $r_1/2$ is odd and thus instead of (238) we have

$$Q_{72oo}^{(1)}(v + \omega_1/2) = \mathrm{e}^{N\pi\, \mathrm{i} r_1 r_2/4}\, R S Q_{72oo}^{(1)}(v) \tag{243}$$

and therefore

$$\left[ Q_{72oo}^{(1)}(v), R S \right] = 0 \tag{244}$$

which has been directly shown in section 5.

It follows from the quasiperiodicity relations (243) and (231) that the eigenvalues of $Q_{72oo}^{(1)}(v)$ are of the form (240) where the sum rule (241) is replaced by

$$\mathrm{e}^{N\pi\, \mathrm{i} r_1 r_2/4} (-1)^{v_{RS}} = (-1)^v, \tag{245}$$

where $(-1)^{v_{RS}}$ are the eigenvalues of $R S$.

## 7.2. Quasiperiodicity for $Q_{72ee}^{(2)}(v; t)$

When $m_1$ and $m_2$ are both even, we found that $Q_{72}^{(1)}$ does not exist and that to solve the $TQ$ equation we needed to use the matrix $Q_{72}^{(2)}(v; t)$ constructed in section 6. This matrix exists only for $N$ even and we recall that for $m_1$ and $m_2$ both even that $r_0$ is even. In fact we will see that for $t = (n + 1/2)\eta$ there will be different cases for $r_0 \equiv 0 \pmod 4$ and $r_0 \equiv 2 \pmod 4$.

We find from (78) and (28)–(31) that the quasiperiodicity properties of $S_R^{(2)}(v)$ are

$$S_R^{(2)}(\alpha, \beta)_{j,k}(v + \omega_1) = (-\alpha)^{r_1}(-1)^{r_1 r_2} S_R^{(2)}(\alpha, \beta)_{j,k}(v), \tag{246}$$

$$S_R^{(2)}(\alpha, \beta)_{k,k+1}(v + \omega_2) = (-\alpha)^b(-1)^{ab} q'^{-1} \, e^{-2\pi i(v - 2k\eta - t - K)/\omega_1} S_R^{(2)}(\alpha, \beta)_{k,k+1}(v), \tag{247}$$

$$S_R^{(2)}(\alpha, \beta)_{k+1,k}(v + \omega_2) = (-\alpha)^b(-1)^{ab} q'^{-1} \, e^{-2\pi i(v + 2k\eta + t - K)/\omega_1} S_R^{(2)}(\alpha, \beta)_{k+1,k}(v). \tag{248}$$

From (246) we find for all $t$ that $Q_R^{(2)}(v; t)$ has the periodicity property (recalling that $N$ is even)

$$Q_R^{(2)}(v + \omega_1; t) = (-S)^{r_1} Q_R^{(2)}(v; t). \tag{249}$$

However, the quasiperiodicity of $Q_R^{(2)}(v; t)$ under $v \to v + \omega_2$ is different for the two cases $t = n\eta$ and $t = (n + 1/2)\eta$ and will be treated separately.

### 7.2.1. Quasiperiodicity for $t = n\eta$.

When $t = n\eta$, the matrix $S^{(2)}(\alpha, \beta)(v + \omega_2)$ may be written as

$$S_R^{(2)}(\alpha, \beta)(v + \omega_2) = (-1)^{nr_0/2}(-\alpha)^b(-1)^{ab} q'^{-1} \, e^{-2\pi i(v - K)/\omega_1} M^{(2;0)} S_R^{(2)}(\alpha, \beta)(v) M^{(2;0)-1} \tag{250}$$

with $M^{(2;0)}$ given by

$$M_{k,k'}^{(2;0)} = \delta_{k,k'} \, e^{-\pi i r_0 k(k-1)/(2L)} (-1)^{nr_0 k/2} \, e^{-\pi i n r_0 k/(2L)} \tag{251}$$

and thus we obtain from (38)

$$Q_R^{(2)}(v + \omega_2; n\eta) = (-S)^b q'^{-N} \, e^{-2\pi i N(v - K)/\omega_1} Q_R^{(2)}(v; t). \tag{252}$$

Thus, for $t = n\eta$, we find from the definition (51) of $Q_{72ee}^{(2)}(v; n\eta)$ that

$$Q_{72ee}^{(2)}(v + \omega_1; n\eta) = (-S)^{r_1} Q_{72ee}^{(2)}(v; n\eta), \tag{253}$$

$$Q_{72ee}^{(2)}(v + \omega_2; n\eta) = (-S)^b q'^{-N} \, e^{-2\pi i N(v - K)/\omega_1} Q_{72ee}^{(2)}(v; n\eta). \tag{254}$$

It follows from (34) that both $b$ and $r_1$ cannot both be even and thus $S$ appears in at least one of (253) or (254) and therefore as previously found in section 6 it follows that $Q_{72ee}^{(2)}(v; n\eta)$ commutes with the operator $S$.

We further conclude from (253) and (254) that the eigenvalues of $Q_{72ee}^{(2)}(v; n\eta)$ may be written in the form

$$Q_{72ee}^{(2)}(v; n\eta) = \mathcal{K} \exp(-i\nu\pi v/\omega_1) \prod_{j=1}^{N} H_m(v - v_j), \tag{255}$$

where the $N$ zeros $v_j$ are in the parallelogram (223) and the sum rules

$$e^{-i\pi\nu} = (-1)^{r_1(v_s - 1)} \tag{256}$$

$$q'^{-\nu} \exp\left(2\pi i \sum_{j=1}^{N} v_j/\omega_1\right) = (-1)^{b(v_s - 1)} \tag{257}$$

are satisfied. In particular, if $r_1$ is even we see from (256) that $\nu = 0$.

### 7.3. Quasiperiodicity for $t = (n + 1/2)\eta$

When $t = (n + 1/2)\eta$, we first consider the case $m_2 \equiv 0 \pmod 4$. In this case, $r_0 \equiv 0 \pmod 4$ and find from (247) and (248) that

$$S_R^{(2)}(\alpha, \beta)(v + \omega_2; (n + 1/2)\eta) = (-1)^{r_0/4}(-\alpha)^b(-1)^{ab}q'^{-1} e^{-2\pi i(v-K)/\omega_1}$$
$$\times M^{(2;1)} S_R^{(2)}(\alpha, \beta)(v; (n + 1/2)\eta) M^{(2;1)-1}, \tag{258}$$

where

$$M_{k,k'}^{(2;1)} = \delta_{k,k'} e^{-\pi i r_0 k(k-1)/(2L)}(-1)^{r_0 k/4} e^{-\pi i(2n+1)r_0 k/(4L)}. \tag{259}$$

Thus, we find from (38) that

$$Q_R^{(2)}(v + \omega_2; (n + 1/2)\eta) = (-S)^b q'^{-N} e^{-2\pi iN(v-K)/\omega_1} Q_R^{(2)}(v; (n + 1/2)\eta). \tag{260}$$

This is identical with (252) for $Q^{(2)}(v; n\eta)$ and thus we conclude that for $r_0 \equiv 0 \pmod 4$ that $Q_{72ee}^{(2)}(v; (n + 1/2)\eta)$ has the same quasiperiodicity properties (253), (254) and form of eigenvalues (255) as does $Q_{72ee}^{(2)}(v; n\eta)$.

We next consider $m_2 \equiv 2 \pmod 4$ where $r_0 \equiv 2 \pmod 4$ $r_1$ is even and $r_2$ is odd. In this case, the similarity transformation in (258) will not exist. The reason for this is that in order for (258) to hold it was necessary that

$$e^{\pi i r_0/4} = \pm 1 \tag{261}$$

which is not the case when $r_0 \equiv 2 \pmod 4$. In this case, the analogous argument shows that $Q_{72ee}^{(2)}(v; (n + 1/2)\eta)$ is quasiperiodic under $v \rightarrow v + 2\omega_2$ and thus has $2N$ zeros in the parallelogram $0, \omega_1, \omega_1 + 2\omega_2, 2\omega_2$.

However, these $2N$ zeros are not independent because there is an additional quasiperiodicity under $v \rightarrow v + \omega_2 + \omega_1/2$. To show this, we use the relations which follow from (A.8)–(A.11) when $r_1$ is even and $r_2$ is odd

$$H_m(v + \omega_1/2 + \omega_2) = -(-1)^{r_1/2+b}(-1)^{r_1 r_2/4+ab}q'^{-1} e^{-2\pi i(v-K)/\omega_1} \Theta_m(v), \tag{262}$$

$$\Theta_m(v + \omega_1/2 + \omega_2) = -(-1)^{r_1 r_2/4+ab}q'^{-1} e^{-2\pi i(v-K)/\omega_1} H_m(v) \tag{263}$$

and find from (78) that

$$S_R^{(2)}(\alpha, \beta)_{k,k+1}(v + \omega_1/2 + \omega_2) = (-\alpha)^{r_1/2+b} f(v) e^{\pi i/2} e^{2\pi i(t+2k\eta)/\omega_1} S_R^{(2)}(-\alpha, \beta)_{k,k+1}(v), \tag{264}$$

$$S_R^{(2)}(\alpha, \beta)_{k+1,k}(v + \omega_1/2 + \omega_2) = (-\alpha)^{r_1/2+b} f(v) e^{-\pi i/2} e^{-2\pi i(t+2k\eta)/\omega_1} S_R^{(2)}(-\alpha, \beta)_{k+1,k}(v), \tag{265}$$

$$S^{(2)}(\alpha, \beta)_{1,L}(v + \omega_1/2 + \omega_2) = (-\alpha)^{r_1/2+b} f(v) e^{-\pi i/2} e^{-2\pi i(t+2L\eta)/\omega_1} S_R^{(2)}(-\alpha, \beta)_{1,L}(v), \tag{266}$$

$$S_R^{(2)}(\alpha, \beta)_{L,1}(v + \omega_1/2 + \omega_2) = (-\alpha)^{r_1/2+b} f(v) e^{\pi i/2} e^{2\pi i(t+2L\eta)/\omega_1} S_R^{(2)}(-\alpha, \beta)_{L,1}(v), \tag{267}$$

where

$$f(v) = -i(-1)^{r_1 r_2/4+ab}q'^{-1} e^{-2\pi i(v-K)/\omega_1}. \tag{268}$$

The expressions (264)–(267) can be written as

$$M^{(2;2)} S_R^{(2)}(\alpha, \beta)(v + \omega_1/2 + \omega_2) M^{(2;2)-1} = \epsilon(-\alpha)^{r_1/2+b} f(v) S_R^{(2)}(-\alpha, \beta)(v) \tag{269}$$

with $\epsilon = \pm 1$ and

$$M_{k,k'}^{(2;2)} = m_k \delta_{k,k'} \tag{270}$$

where

$$m_k = (i\epsilon)^{k-1} e^{2\pi i(k-1)(t+k\eta)/\omega_1} \tag{271}$$

and for consistency we need

$$(i\epsilon)^L e^{2\pi itL/\omega_1} e^{2\pi iL(L+1)\eta/\omega_1} = 1. \tag{272}$$

Using (37) and the fact that $r_0/2$ and $L$ are odd, (272) reduces to

$$(i\epsilon)^L e^{\pi itr_0/(2\eta)} = 1 \tag{273}$$

which with an appropriate choice of $\epsilon = \pm 1$ is satisfied for $t = (n + 1/2)\eta$ as desired. Thus, we obtain from (269) for $m_2 \equiv 2 \pmod 4$ and $N$ even

$$Q_{72ee}^{(2)}(v + \omega_1/2 + \omega_2; (n + 1/2)\eta)$$
$$= (-1)^{N/2} S^{r_1/2+b}(-1)^{Nr_1r_2/4} q'^{-N} e^{-2\pi iN(v-K)/\omega_1} R Q_{72ee}^{(2)}(v; (n + 1/2)\eta). \tag{274}$$

Finally, we recall from (33) that $b$ must be odd because $r_1$ is even and $r_2$ is odd and that because $r_0 \equiv 2 \pmod 4$ we have $r_1 \equiv 2(0) \pmod 4$ for $m_1 \equiv 2(0) \pmod 4$. Thus, we find that for $m_1 \equiv 2 \pmod 4$ and $m_2 \equiv 2 \pmod 4$ that

$$Q_{72ee}^{(2)}(v + \omega_1/2 + \omega_2; (n + 1/2)\eta) = q'^{-N} e^{-2\pi iN(v-K)/\omega_1} R Q_{72ee}^{(2)}(v; (n + 1/2)\eta) \tag{275}$$

from which it follows in agreement with (14) that $Q_{72ee}^{(2)}(v; (n + 1/2)\eta)$ commutes with $R$. For $m_2 \equiv 2 \pmod 4$ and $m_1 \equiv 0 \pmod 4$

$$Q_{72ee}^{(2)}(v + \omega_1/2 + \omega_2; (n + 1/2)\eta)$$
$$= (-1)^{N/2} q'^{-N} e^{-2\pi iN(v-K)/\omega_1} RS Q_{72ee}^{(2)}(v; (n + 1/2)\eta) \tag{276}$$

from which it follows in agreement with (15) that $Q_{72ee}^{(2)}(v; (n + 1/2)\eta)$ commutes with $RS$.

It follows from (275) and (276) that for $m_2 \equiv 2 \pmod 4$ there are $N$ zeros in the fundamental parallelogram

$$0, \omega_1, 3\omega_1/2 + \omega_2, \omega_1/2 + \omega_2. \tag{277}$$

For $m_1 \equiv 2 \pmod 4$ and $m_2 \equiv 2 \pmod 4$ it follows from (275) that the eigenvalues of $Q_{72ee}^{(2)}(v; (n + 1/2)\eta)$ are of the form

$$Q_{72ee}^{(2)}(v; (n + 1/2)\eta) = \mathcal{K} \prod_{j=1}^{N} H_m\left(\frac{v - v_j}{2}\right) H_m\left(\frac{v - v_j + \omega_1/2 + \omega_2}{2}\right)$$
$$\times H_m\left(\frac{v - v_j + \omega_1 + 2\omega_2}{2}\right) H_m\left(\frac{v - v_j + 3\omega_1/2 + 3\omega_2}{2}\right) \tag{278}$$

which as required by the condition (249) is periodic in $v \to v + \omega_1$ and satisfies the quasiperiodicity condition (275) with the sum rule

$$(-1)^{v_R} = q'^{-3N} e^{2\pi i \sum_{j=1}^{N}(v_j+K)/\omega_1} = q'^{-3N-Nr_2} e^{2\pi i \sum_{j=1}^{N} v_j/\omega_1}. \tag{279}$$

For $m_1 \equiv 0 \pmod 4$ and $m_2 \equiv 2 \pmod 4$ it follows from (276) that the eigenvalues of $Q_{72ee}^{(2)}(v; (n + 1/2)\eta)$ are of the form (278) with the sum rule

$$(-1)^{N/2}(-1)^{v_{RS}} = q'^{-3N} e^{2\pi i \sum_{j=1}^{N}(v_j+K)/\omega_1} = q'^{-3N-Nr_2} e^{2\pi i \sum_{j=1}^{N} v_j/\omega_1}. \tag{280}$$

## 8. Conclusion

In this paper, we have extended the construction of the matrices $Q_{72}(v)$ that solve the $TQ$ equation (1)–(3) which were introduced in [1] to solve the eight-vertex model in the elliptic root of unity case (4) from $m_2 = 0$ to $m_2 \neq 0$ and have found that the constructions depend on the parities of $m_1$ and $m_2$. In all cases, we have examined the matrices $Q_{72}(v)$ are nondegenerate.

We have found that in all cases the matrix $Q_{72}(v)$ defined at roots of unity (4) commutes with only one of the three matrices $S$, $R$ and $RS$ which are the discrete symmetries of the transfer matrix $T(v)$ of the eight-vertex model. This is in contrast with the matrix $Q_{73}(v)$ of [2] which commutes with all three of the discrete symmetry operators. This failure of $Q_{72}(v)$ to commute with one of the discrete symmetry operators of $T(v)$ when combined with the nondegeneracy of $Q_{72}(v)$ provides an explanation of the existence of some eigenvalues of $T(v)$ which are at least doubly degenerate. This mechanism is not possible for $Q_{73}(v)$ at roots of unity and is an indication that there is a sense in which $Q_{72}(v)$ contains information which is lacking in $Q_{73}(v)$.

Perhaps the most novel feature of our results is that in the case where both $m_1$ and $m_2$ are even that there are different cases depending on whether or not $m_1$ and $m_2$ are divisible by four and that for $m_2 \equiv 2(\mathrm{mod}\,4)$ there exist two matrices with different commutation properties with the discrete symmetry operators. These matrices map a degenerate subspace of $T$ on another degenerate subspace with opposite eigenvalue of the discrete symmetry operator thereby doubling the size of degenerate multiplets of T.

We finally note that even though there are cases where $Q_R^{(1)}(v)$ or $Q_R^{(2)}(v)$ obey the interchange relation (49) with all four of the operators $I$, $S$, $R$ and $RS$, there is in fact no matrix $Q_{72}(v)$ which shares with $Q_{73}(v)$ the property of commuting with all three operators $R$, $S$ and $RS$. This would happen for $Q_R^{(1)}(v)$ if $m_1$ and $m_2$ are even[3] and for $Q_R^{(2)}(v;t)$ if $m_1 \equiv 2(\mathrm{mod}\,4)$ and $m_2 \equiv 0(\mathrm{mod}\,4)$ and $t = (n+1/2)\eta$,[4] but in these cases $Q_R(v)$ is singular and thus the construction (51) of $Q_{72}(v)$ cannot be made.

The various new properties of $Q_{72}(v)$ found in this paper for $m_2 \neq 0$ must contain useful information about the still undetermined symmetry algebra of the eight-vertex model at elliptic roots of unity.

## Appendix A. Properties of the modified theta functions

The functions $\mathrm{H}(v)$ and $\Theta(v)$ defined by (22) and (23) have the following well-known properties

$$\mathrm{H}(-v) = -\mathrm{H}(v), \qquad \Theta(-v) = \Theta(v) \tag{A.1}$$

$$\mathrm{H}(v + 2nK) = (-)^n \mathrm{H}(v), \qquad \Theta(v + 2nK) = \Theta(v) \tag{A.2}$$

$$\mathrm{H}(v + 2\mathrm{i}nK') = (-1)^n q^{-n^2} e^{-n\pi\,\mathrm{i}v/K} \mathrm{H}(v) \tag{A.3}$$

$$\Theta(v + 2\mathrm{i}nK') = (-1)^n q^{-n^2} e^{-n\pi\,\mathrm{i}v/K} \Theta(v) \tag{A.4}$$

$$\Theta(v + \mathrm{i}K') = \mathrm{i}q^{-1/4} e^{-\frac{\pi\,\mathrm{i}v}{2K}} \mathrm{H}(v), \qquad \mathrm{H}(v + \mathrm{i}K') = \mathrm{i}q^{-1/4} e^{-\frac{\pi\,\mathrm{i}v}{2K}} \Theta(v). \tag{A.5}$$

It follows immediately from (A.1)–(A.5) that the modified theta functions $\mathrm{H}_m(v)$ and $\Theta_m(v)$ defined by (25) have the properties

$$\mathrm{H}_m(2K - v) = \mathrm{H}_m(v), \qquad \Theta_m(2K - v) = \Theta_m(v), \tag{A.6}$$

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[3] See table 1.
[4] See table 3.

$$H_m(-v) = -\exp\left(\frac{i\pi m_2 v}{2L\eta}\right) H_m(v), \qquad \Theta_m(-v) = \exp\left(\frac{i\pi m_2 v}{2L\eta}\right) \Theta_m(v), \qquad \text{(A.7)}$$

$$H_m(u + 2rK + 2isK') = (-1)^r(-1)^{rs} H_m(u)$$
$$\times \exp\{(\pi i(rm_2/2 - sm_1)[u + (r-1)K + isK']/(L\eta)\} \qquad \text{(A.8)}$$

and

$$\Theta_m(u + 2rK + 2isK')$$
$$= (-1)^{rs}\Theta_m(u)\exp\{(\pi i(rm_2/2 - sm_1)[u + (r-1)K + isK']/(L\eta)\}, \qquad \text{(A.9)}$$

$$\Theta_m(v + iK') = iq^{-1/4}\exp\left(\frac{-i\pi m_1 v}{2L\eta}\right) C\,H_m(v),$$

$$H_m(v + iK') = iq^{-1/4}\exp\left(\frac{-i\pi m_1 v}{2L\eta}\right) C\Theta_m(v),$$

$$\text{(A.10)}$$

where

$$C = \exp\left(\frac{\pi m_2 K'}{8KL\eta}(2K - iK')\right). \qquad \text{(A.11)}$$

For convenience we note the special cases

$$H_m(v + 2L\eta) = (-1)^{m_1} i^{m_1 m_2} \begin{cases} H_m(v) & \text{if } m_2 = \text{even} \\ \Theta_m(v) & \text{if } m_2 = \text{odd,} \end{cases} \qquad \text{(A.12)}$$

$$\Theta_m(v + 2L\eta) = i^{m_1 m_2} \begin{cases} \Theta_m(v) & \text{if } m_2 = \text{even} \\ H_m(v) & \text{if } m_2 = \text{odd.} \end{cases} \qquad \text{(A.13)}$$

The properties (A.12), (A.13), (28)–(31) follow immediately from (A.8)–(A.11).

The functions $H_m(v)$ and $\Theta_m(v)$ satisfy the identities

$$H_m(u)\,H_m(v)\,H_m(w)\,H_m(u + v + w) + \Theta_m(u)\Theta_m(v)\Theta_m(w)\Theta_m(u + v + w)$$
$$= \Theta_m(0)\Theta_m(u + v)\Theta_m(u + w)\Theta_m(v + w), \qquad \text{(A.14)}$$

$$H_m(u)\,H_m(v)\Theta_m(w)\Theta_m(u + v + w) + \Theta_m(u)\Theta_m(v)\,H_m(w)\,H_m(u + v + w)$$
$$= \Theta_m(0)\Theta_m(u + v)\,H_m(u + w)\,H_m(v + w). \qquad \text{(A.15)}$$

## Appendix B. Modified theta functions and Jacobi theta functions

We write the modified theta functions

$$H_m(u) = \exp\left(\frac{i\pi m_2(u - K)^2}{8KL\eta}\right) H(u), \qquad \Theta_m(u) = \exp\left(\frac{i\pi m_2(u - K)^2}{8KL\eta}\right) \Theta(u) \quad \text{(B.1)}$$

in terms of theta functions with characteristics defined as

$$\Theta_{\epsilon\epsilon'} = \sum_{-\infty}^{\infty} q^{(n+\epsilon/2)^2} \exp(2\pi i(n + \epsilon/2)(z + \epsilon'/2)), \qquad \text{(B.2)}$$

$$q = \exp(i\pi\tau). \qquad \text{(B.3)}$$

To cancel a common divisor of $m_1$ and $m_2$ in

$$2L\eta = 2m_1 K + im_2 K', \qquad \text{(B.4)}$$

we define

$$(2m_1, m_2) = r_0, \qquad 2m_1 = r_0 r_1, \qquad m_2 = r_0 r_2, \tag{B.5}$$

$$2L\eta = r_0(r_1 K + i r_2 K'). \tag{B.6}$$

It follows

$$H_m(u) = -\exp\left(\frac{i\pi r_2 w^2}{r_1 + r_2 \tau}\right) \Theta_{11}(w + 1/2), \tag{B.7}$$

$$\Theta_m(u) = \exp\left(\frac{i\pi r_2 w^2}{r_1 + r_2 \tau}\right) \Theta_{01}(w + 1/2), \tag{B.8}$$

where

$$w = u/2K - 1/2, \tag{B.9}$$

or after a shift of the argument

$$H_m(u) = \exp\left(\frac{i\pi r_2 w^2}{r_1 + r_2 \tau}\right) \Theta_{10}(w), \tag{B.10}$$

$$\Theta_m(u) = \exp\left(\frac{i\pi r_2 w^2}{r_1 + r_2 \tau}\right) \Theta_{00}(w). \tag{B.11}$$

The second step is to use the functional relation of theta functions. From [22]

$$\exp\left(\frac{i\pi c w^2}{c\tau + d}\right) \Theta_{\tilde{\epsilon}, \tilde{\epsilon}'}(w) = \kappa^{-1}(\epsilon, \epsilon', \gamma)(c\tau + d)^{-1/2} \Theta_{\epsilon, \epsilon'}\left(\frac{w}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right), \tag{B.12}$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1 \tag{B.13}$$

and

$$\tilde{\epsilon} = a\epsilon + c\epsilon' - ac, \qquad \tilde{\epsilon}' = b\epsilon + d\epsilon' + bd. \tag{B.14}$$

We use (B.12) to derive from (B.10) and (B.11) for $c = r_2$ and $d = r_1$

$$H_m(u) = \kappa(\epsilon_1, \epsilon_1', \gamma)^{-1}(r_1 + r_2 \tau)^{-1/2} \Theta_{\epsilon_1, \epsilon_1'}\left(\frac{w}{r_1 + r_2 \tau}, \tau'\right), \tag{B.15}$$

$$\Theta_m(u) = \kappa(\epsilon_2, \epsilon_2', \gamma)^{-1}(r_1 + r_2 \tau)^{-1/2} \Theta_{\epsilon_2, \epsilon_2'}\left(\frac{w}{r_1 + r_2 \tau}, \tau'\right), \tag{B.16}$$

$$\epsilon_1 = a r_1 (r_1 + r_2), \qquad \epsilon_1' = -b(1 + a(r_1 + r_2)), \tag{B.17}$$

$$\epsilon_2 = r_1 r_2 (a + b), \qquad \epsilon_2' = -ab(r_1 + r_2), \tag{B.18}$$

where

$$\tau' = \frac{a\tau + b}{r_2 \tau + r_1}, \qquad w = \frac{u - K}{2K} \tag{B.19}$$

and where the integers $a, b$ are solutions of

$$a r_1 - b r_2 = 1. \tag{B.20}$$

The indices $\epsilon_1, \ldots, \epsilon_2'$ can be shifted to 0, 1 such that the theta functions on the right-hand sides of (B.15), (B.16) become

$$\Theta_{00}, \Theta_{01}, \Theta_{10}, \Theta_{11} \equiv \Theta_1, \Theta, H_1, H.$$

It follows from equations (B.15) and (B.16) that the period and quasiperiod of $H_m(u)$ and $\Theta_m(u)$ are

$$\omega_1 = 2(r_1 K + i r_2 K'), \qquad \omega_2 = 2(bK + iaK'). \tag{B.21}$$

The coefficient $\kappa(\epsilon, \epsilon', \gamma)$ is not used in this paper. It is an eighth root of unity and its dependence on $\epsilon, \epsilon'$ is given by [22]

$$\kappa(\epsilon, \epsilon', \gamma) = \kappa(0, 0, \gamma) \exp\left(\frac{-i\pi}{4}(\epsilon^2 ab + \epsilon'^2 cd + 2\epsilon\epsilon' bc + 2(a\epsilon + c\epsilon')bd)\right). \tag{B.22}$$

## Appendix C. The equation for $Q_R^{(2)}(v; t)$

We establish here the relation (79). Using the method of [1] and [16], we write

$$T(v)Q_R^{(2)}(v; t) = \operatorname{Tr} A(\alpha_1, \beta_1) \cdots A(\alpha_N, \beta_N) + \operatorname{Tr} B(\alpha_1, \beta_1) \cdots B(\alpha_N, \beta_N), \tag{C.1}$$

where $A(\alpha, \beta)$ and $B(\alpha, \beta)$ are $2L \times 2L$ matrices given by

$$\begin{pmatrix} A(\alpha, \beta) & 0 \\ C(\alpha, \beta) & B(\alpha, \beta) \end{pmatrix} = \begin{pmatrix} I & -P \\ 0 & I \end{pmatrix} U(\alpha, \beta) \begin{pmatrix} I & P \\ 0 & I \end{pmatrix} \tag{C.2}$$

with

$$P_{m,n} = p_n \delta_{m,n} \tag{C.3}$$

and $p_n$ given by (77) and where

$$U(+, \beta) = \begin{pmatrix} a S_R^{(2)}(+, \beta) & d S^{(2)}(-, \beta) \\ c S_R^{(2)}(-, \beta) & b S_R^{(2)}(+, \beta) \end{pmatrix}, \tag{C.4}$$

$$U(-, \beta) = \begin{pmatrix} b S_R^{(2)}(-, \beta) & c S^{(2)}(+, \beta) \\ d S_R^{(2)}(+, \beta) & a S_R^{(2)}(-, \beta) \end{pmatrix}. \tag{C.5}$$

Then, from (C.2) and (78) we use

$$H_m(-v)/\Theta_m(-v) = -H_m(v)/\Theta_m(v) \tag{C.6}$$

and the identities (A.14), (A.15) to find

$$A(+, \beta)_{k,k+1} = -\tau_{\beta,-k} \frac{\Theta_m(-(2k+1)\eta - t)}{\Theta_m(-(2k-1)\eta - t)} h(v - \eta) H_m(v - 2k\eta + 2\eta - t) \tag{C.7}$$

$$A(+, \beta)_{k+1,k} = \tau_{\beta,k} \frac{\Theta((2k-1)\eta + t)}{\Theta((2k+1)\eta + t)} h(v - \eta) H_m(v + 2k\eta + 2\eta + t) \tag{C.8}$$

$$A(-, \beta)_{k,k+1} = \tau_{\beta,-k} \frac{\Theta_m(-(2k+1)\eta - t)}{\Theta_m(-(2k-1)\eta - t)} h(v - \eta) \Theta_m(v - 2k\eta + 2\eta - t) \tag{C.9}$$

$$A(-, \beta)_{k+1,k} = \tau_{\beta,k} \frac{\Theta_m((2k-1)\eta + t)}{\Theta_m((2k+1)\eta + t)} h(v - \eta) \Theta_m(v + 2k\eta + 2\eta + t) \tag{C.10}$$

$$B(+, \beta)_{k,k+1} = -\tau_{\beta,-k} \frac{\Theta_m(-(2k-1)\eta - t)}{\Theta_m(-(2k+1)\eta - t)} h(v + \eta) H_m(v - 2k\eta - 2\eta - t) \tag{C.11}$$

$$B(+, \beta)_{k+1,k} = \tau_{\beta,k} \frac{\Theta_m((2k+1)\eta + t)}{\Theta_m((2k-1)\eta + t)} h(v+\eta) \, H_m(v + 2k\eta - 2\eta + t) \tag{C.12}$$

$$B(-, \beta)_{k,k+1} = \tau_{\beta,-k} \frac{\Theta_m(-(2k-1)\eta - t)}{\Theta_m(-(2k+1)\eta - t)} h(v+\eta) \Theta_m(v - 2k\eta - 2\eta - t) \tag{C.13}$$

$$B(-, \beta)_{k+1,k} = \tau_{\beta,k} \frac{\Theta_m((2k+1)\eta + t)}{\Theta_m((2k-1)\eta + t)} h(v+\eta) \Theta_m(v + 2k\eta - 2\eta + t). \tag{C.14}$$

From (A.7) we have

$$\frac{\Theta_m(-(2k+1)\eta - t)}{\Theta_m(-(2k-1)\eta - t)} = \exp(\pi \, \mathrm{i} m_2/L) \frac{\Theta_m((2k+1)\eta + t)}{\Theta_m((2k-1)\eta + t)} \tag{C.15}$$

and thus with the definition (80) of $\omega$ and

$$f_k = \omega \frac{\Theta((2k+1)\eta + t)}{\Theta((2k-1)\eta + t)}, \tag{C.16}$$

we may write (C.7)–(C.14) as

$$A(+, \beta)_{k,k+1}(v) = \omega f_k h(v-\eta) S_R(+, \beta)_{k,k+1}(v + 2\eta) \tag{C.17}$$

$$A(+, \beta)_{k+1,k}(v) = \omega f_k^{-1} h(v-\eta) S_R(+, \beta)_{k+1,k}(v + 2\eta) \tag{C.18}$$

$$A(-, \beta)_{k,k+1}(v) = \omega f_k h(v-\eta) S_R(-, \beta)_{k,k+1}(v + 2\eta) \tag{C.19}$$

$$A(-, \beta)_{k+1,k}(v) = \omega f_k^{-1} h(v-\eta) S_R(-, \beta)_{k+1,k}(v + 2\eta) \tag{C.20}$$

$$B(+, \beta)_{k,k+1}(v) = \omega^{-1} f_k^{-1} h(v+\eta) S_R(+, \beta)_{k,k+1}(v - 2\eta) \tag{C.21}$$

$$B(+, \beta)_{k+1,k}(v) = \omega^{-1} f_k h(v+\eta) S_R(+, \beta)_{k+1,k}(v - 2\eta) \tag{C.22}$$

$$B(-, \beta)_{k,k+1}(v) = \omega^{-1} f_k^{-1} h(v+\eta) S_R(-, \beta)_{k,k+1}(v - 2\eta) \tag{C.23}$$

$$B(-, \beta)_{k+1,k}(v) = \omega^{-1} f_k h(v+\eta) S_R(-, \beta)_{k+1,k}(v - 2\eta). \tag{C.24}$$

The $T Q_R$ equation (C.1) will be obtained if the factors of $f_k$ can be removed by a diagonal similarity transformation

$$S_A A(\alpha, \beta) S_A^{-1} \tag{C.25}$$

with

$$S_{A;k,k'} = s_k \delta_{k,k'}, \tag{C.26}$$

this is accomplished for the elements $A_{k,k+1}(\alpha, \beta)$ and $A_{k+1,k}(\alpha, \beta)$ with $1 \leqslant k, \leqslant L - 1$

$$\frac{s_k f_k}{s_{k+1}} = \frac{s_{k+1}}{s_k f_k} = \frac{s_1}{s_L f_L} = \frac{s_L f_L}{s_1} \qquad \text{for} \qquad 1 \leqslant k \leqslant L - 1. \tag{C.27}$$

From the first equation in (C.27), we have

$$\frac{s_k f_k}{s_{k+1}} = \pm 1, \tag{C.28}$$

where the choice $\pm 1$ is still to be determined and from (C.28) we have

$$s_k = (\pm 1)^{k-1} s_1 \omega^{k-1} \frac{\Theta_m[(2k-1)\eta + t]}{\Theta_m(\eta + t)}. \tag{C.29}$$

The remaining equations in (C.28) will hold if

$$\frac{s_L f_L}{s_1} = \pm 1 \tag{C.30}$$

and using (C.29) we obtain

$$(\pm 1)^L \omega^L \frac{\theta_m[(2L+1)\eta + t]}{\theta_m(\eta + t)} = 1 \tag{C.31}$$

which if we further use (A.13) restricted to the present case where $m_1$ and $m_2$ are even and $L$ is odd determines that the factor $\pm 1$ is

$$\pm 1 = (-1)^{m_2/2}. \tag{C.32}$$

An identical computation holds for the matrices $B(\alpha, \beta)$ and thus (recalling the $N$ is even) we have proven that (79) holds.

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